What is Zeckendorf's Theorem?

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Abstract

While Fibonacci numbers can quite easily be classified as a complete sequence, they have the unusual property that a particular explicitly defined subset of such sequences is complete and also gives unique representations of natural numbers. This can be extended to all integers via the "nega-Fibonacci" numbers. Further, an unusual binary operation on the natural numbers may be defined and proven to be associative.

1 Introduction and Background

Edouard Zeckendorf (1901-1983) was a Belgian mathematician (and doctor and army officer). He published [4] in 1972, though his work was not original, in accordance with Stigler's Law of Eponymy. 20 years prior to Zeckendorf's publication, Dutch mathematician Gerrit Lekkerkerker had already published [2] using similar techniques.

2 Preliminary Results

Definition 1: The n-th *Fibonacci number* F_n is defined recursively by the equation $F_n = F_{n-1} + F_{n-2}$, where $F_0 = 0$ and $F_1 = 1$.

Note: We will say that Fibonacci numbers F_n and F_m are *consecutive* when $n = m \pm 1$, rather than when $F_n = F_m \pm 1$.

Lemma 1: For any increasing sequence $(c_i)_{i=0}^{k}$ such that $c_i \geq 2$ and $c_{i+1} > c_i + 1$ for $i \geq 0$, we have

$$\sum_{i=0}^{k} F_{c_i} < F_{c_k+1}.$$

Proof. We note that, taking F_3 to be the largest term, the largest sum of nonconsecutive Fibonacci numbers is

$$F_3 + F_1 = 3 + 1 = 4 < 5 = F_4.$$

Assume that sums of nonconsecutive Fibonacci numbers up to F_{j-1} are all less that F_j. Such a sum up to F_j will contain no F_{j-1}, and will thus be a sum of F_j and a nonconsecutive sum of Fibonacci numbers up F_{j-2}, which will be smaller than F_{j-1} by the inductive hypothesis. Thus, for any sequence (c_i) as described above where $c_k = j$, we have,

$$\sum_{i=0}^{k} F_{c_i} = F_j + \sum_{i=0}^{k-1} F_{c_i} < F_j + F_{j-1} = F_{j+1}.$$

This proves the result for all $j \in \mathbb{N}$ inductively. □

Corollary 1: Any sum of Fibonacci numbers up to F_n is less than F_{n+2}.

Proof. Let $n = c_k$. Using the lemma after splitting the sum into sum of even and odd index Fibonacci numbers, we see that, if c_e is the largest even index and c_o the largest odd,

$$\sum_{i=0}^{k} F_{c_i} = \sum_{c_i \text{ odd}} F_{c_i} + \sum_{c_i \text{ even}} F_{c_i} < F_{c_o+1} + F_{c_e+1} \leq F_n + F_{n-1} = F_{n+2}.$$

This could have easily be proven directly with induction, but is shorter with the lemma, which handles the induction for us. □

3 Zeckendorf's Theorem

Theorem 1: (*Zeckendorf's Theorem*) Let n be a positive integer. Then there is a unique increasing sequence $(c_i)_{i=0}^k$ such that $c_i \geq 2$ and $c_{i+1} > c_i + 1$ for $i \geq 0$, and that

$$n = \sum_{i=0}^{k} F_{c_i}.$$

We will call such a sum the *Zeckendorf representation* for n.

Proof. We begin with a proof of existence. We see that $1 = F_2$, $2 = F_3$, $3 = F_4$, and $4 = F_2 + F_4 = 1 + 3$. Suppose now that we can find such a representation for all positive integers up to k. If $k + 1$ is a Fibonacci number, then that provides the Zeckendorf representation. If $k + 1$ is not a Fibonacci number, then $\exists j \in \mathbb{N}$ such that $F_j < k+1 < F_{j+1}$. Define $a = k+1-F_j$, so $a \leq k$, meaning a has a Zeckendorf representation by hypothesis. We also note that

$$F_j + a = k + 1 < F_{j+1} = F_j + F_{j-1}$$
$$a < F_{j-1}.$$

Thus, the Zeckendorf representation of a does not contain a F_{j-1} term, so $k+1 = F_j + a$ will yield a Zeckendorf representation for $k+1$. This proves the existence of the Zeckendorf representation for positive integers by induction.

We will now prove uniqueness. Let n be a positive integer with two nonempty sets of terms S and T which form Zeckendorf representations of n. Let $S' = S \setminus T$ and $T' = T \setminus S$. Since both sets lost the same common elements, we still have

$$\sum_{x \in S} x - \sum_{a \in S \cap T} a = \sum_{y \in T} y - \sum_{b \in S \cap T} b$$
$$\sum_{x \in S'} x = \sum_{y \in T'} y.$$

Thus, if either S' or T' is empty, it will yield a sum of 0. Since all terms are non-negative, the other sum, equaling 0, must also be empty, meaning that $S' = T' = \varnothing$, so $S = S \cap T = T$.

Let us now assume that both sets S' and T' are nonempty. Let $F_s = \max S'$ and $F_t = \max T'$. Since $S' \neq T'$, we may say without loss of generality that $F_s < F_t$. By the lemma, we may say that

$$\sum_{x \in S'} x < F_{s+1} \leq F_t.$$

Since the sums over S' and T' are non-negative and equal, this is a contradiction, so $S' = T' = \varnothing$, and $S = T$. □

While uniqueness can perhaps be more easily directly proven, this contradiction is more in the traditional spirit.

4 Fibonacci Multiplication

Definition 2: We will define the operation of *Fibonacci multiplication* $\circ : \mathbb{N}^2 \to \mathbb{N}$ as follows: Given two numbers with Zeckendorf representations

$$a = \sum_{i=0}^{k} F_{c_i}, \qquad b = \sum_{j=0}^{l} F_{d_j},$$

we will define

$$a \circ b = \sum_{i=0}^{k} \sum_{j=0}^{l} F_{c_i + d_j}.$$

Theorem 2: $(\mathbb{N}, \circ)$ forms a (commutative) semigroup.

Proving commutativity of $\circ$ over $\mathbb{N}$ is a simple rearrangement of finite sums. Were the circle product to always immediately yield a Zeckendorf representation of the product, associativity would be easy, but this can quickly be seen to not be the case. Donald Knuth (1938-) published his proof of associativity of Fibonacci multiplication in 1988 in [1]. Our proof will follow his technique, which rephrases the problem through a Fibonacci variation on the usual power-based radix notation. Essentially, when we concatenate digits this way, we mean

$$n = (d_s d_{s-1} \dots d_1 d_0)_F = \sum_{i=0}^{s} d_i F_i.$$

It is reasonably clear to see the relation to Zeckendorf representations.

Proposition 1: We have that $(d_s d_{s-1} \dots d_1 d_0)_F$ is a Zeckendorf representation precisely when

(Z1) $d_i = 0$ or $d_i = 1$, $\forall i \in \mathbb{N}$,

(Z2) $d_i d_{i+1} = 0$, $\forall i \in \mathbb{N}$,

(Z3) $d_0 = d_1 = 0$,

are all satisfied.

A simple process to add 1 in radix-F gives another existence proof for Zeckendorf's Theorem:

Theorem 3: Every positive integer has a Zeckendorf representation.

Proof. We first note that $1 = (100)_F$ satisfies (Z1-3). We see that, if we have a Zeckendorf representation of n, then

$$\begin{aligned} n &= (d_s \dots d_2 d_1 d_0)_F \\ &= (d_s \dots d_2 00)_F \\ n+1 &= (d_s \dots d_2 11)_F \\ &= (d_s \dots d_2 10)_F. \end{aligned}$$

To turn our representation of $n+1$ into a Zeckendorf representation, we may repeatedly change the strings of digits via the following replacement rule:

$$0\,1\,1 \rightarrow 1\,0\,0$$

which preserves the value of $(d_s \dots d_2 d_1 d_0)_F$. If $d_2 = 1$ for $n+1$, then choose $d_1 = 1$ and $d_0 = 0$. Otherwise, choose $d_1 = d_0 = 1$. We will never see the string $(\dots 111 \dots)_F$ thanks to (Z2) for n. We notice that the rule clearly preserves (Z1), and once no more iterations of the replacement can be made, we will have (Z2). The first replacement immediately handles (Z3). The last issue is a potential cycle under the replacement, but this is impossible, since the replacement increases the value of $(d_s \dots d_2 d_1 d_0)_2$ with each iteration, so the process must eventually terminate, as a number has only finitely many representations in radix-F. Hence, we will eventually have (Z1), (Z2), and (Z3), and thus, a Zeckendorf representation for $n+1$. □

We may further generalize the addition of two numbers in radix notation. To perform such an addition, we may simply add two numbers digit-wise. To take any arbitrary representation $(d_s \dots d_1 d_0)_F$ and turn it into a Zeckendorf representation, we may carry out the value-preserving replacements

$$0\,(d+1)\,(e+1) \to 1\,d\,e \tag{1}$$

$$0\,(d+2)\,0\,e \to 1\,d\,0\,(e+1) \tag{2}$$

from left to right until we can no longer do so.

Lemma 2: If $d_i \leq 2$ for $i \geq 2$ and $d_1 = d_2 = 0$, then these two replacement rules may be used to turn $(d_s \dots d_1 d_0)_F$ into $(d'_t \dots d'_2 d'_1 d'_0)_F$ which satisfies (Z1) and (Z2).

Proof. We have vacuous truth when $s \leq 1$. When $s > 1$, we may apply the carrying process to $(d_s \dots d_1 d_0)_F$ repeatedly, and since both the rules (1) and (2) increase $(d_s \dots d_1 d_0)_2$, the process must terminate. Since (1) can't be applied, we must have (Z2). Since (2) cannot be applied, we must have (Z1). This will yield the representation $(d'_t \dots d'_2 d'_1 d'_0)_F$. □

The last remaining issue is to ensure (Z3). If an addition does not ever "carry down" into d_1 or d_0 via (2), we will call it *clean*. Let $\overline{n}$ be the number of trailing zeroes in the radix-F notation for the Zeckendorf representation of n.

Lemma 3: We have $\overline{m+n} \geq \min(\overline{m}, \overline{n}) - 2$ for $m, n \in \mathbb{N}$.

Proof. Carrying out the process in lemma 2 can be seen to only move into the first two trailing zeroes, if the process is shifted left accordingly. □

It is fairly plain to see that circle multiplication with radix-F notation is totally analogous to ordinary multiplication with binary. When performing a multiplication

$$a \circ b = \sum_{i=0}^{k} \sum_{j=0}^{l} F_{c_i + d_j} = \sum_{j=0}^{l} a \circ F_{d_j},$$

we will call each $a \circ F_{d_j}$ a partial product of $a \circ b$. A quick exercise is to use radix-F notation to show that circle multiplication is monotonic, namely, that, for $l, m, n \in \mathbb{N}$,

$$l < m \implies l \circ n < m \circ n.$$

Lemma 4: Radix-F addition of the partial products of $m \circ n$ is clean.

Proof. We may observe that, for the partial product $m \circ F_k$, we have

$$\overline{m \circ F_k} = \overline{m} + k \geq k + 2.$$

Let $n = \sum_{c=0}^{r} F_{k_c}$ be the Zeckendorf representation. Since $k_{c+1} \geq k_c + 2$, we see that, by lemma 3

$$\begin{aligned}
\overline{m \circ F_{k_r} + m \circ F_{k_{r-1}}} &\geq \min\left(\overline{m \circ F_{k_r}}, \overline{m \circ F_{k_{r-1}}}\right) - 2 \\
&\geq k_{r-1} + 2 - 2 \\
&= k_{r-1} \\
\overline{\sum_{c=i}^{r} m \circ F_{k_c}} &\geq \min\left(\overline{\sum_{c=i+1}^{r} m \circ F_{k_c}}, \overline{m \circ F_{k_i}}\right) - 2 \\
&\geq k_i + 2 - 2 \\
&= k_i.
\end{aligned}$$

Repeatedly applying this process,

$$\overline{m \circ n} = \overline{\sum_{b=0}^{q} \sum_{c=0}^{r} F_{j_b + k_c}} = \overline{\sum_{c=0}^{r} m \circ F_{k_c}} \geq k_0.$$

Since $k_1 \geq 2$, we see that our addition was clean. □

Theorem 4: Define the Zeckendorf representations

$$l = \sum_{a=0}^{p} F_{i_a}, \qquad m = \sum_{b=0}^{q} F_{j_b}, \qquad n = \sum_{c=0}^{r} F_{k_c}.$$

Then we have

$$(l \circ m) \circ n = \sum_{a=0}^{p} \sum_{b=0}^{q} \sum_{c=0}^{r} F_{i_a + j_b + k_c}.$$

Proof. We may cleanly obtain the partial products of $l \circ m$ shifted left k, summing $(l \circ F_j) \circ F_k$. This gives

$$(l \circ m) \circ F_k = \sum_{a=0}^{p} \sum_{b=0}^{q} F_{i_a + j_b + k}.$$

and we get the desired result by cleanly summing over k via lemma 4. □

This proves Theorem 2 by the symmetry of $(l \circ m) \circ n$. The proof of lemma 4 can be extended to make the resulting sum from Theorem 4 clean, allowing for arbitrary groupings of arbitrarily many Fibonacci products.

Knuth claimed that his work was inspired by [3]. In their paper, Porta and Stolarksy proved the associativity of $* : \mathbb{N}^2 \to \mathbb{N}$, defining

$$m * n = mn + \lfloor \phi m \rfloor \lfloor \phi n \rfloor,$$

where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden mean. Thus, $(\mathbb{N}, \circ)$ and $(\mathbb{N}, *)$ both form commutative semigroups.

5 Other Results

As it turns out, we can extend Zeckendorf's Theorem to all integers using the so-called "nega-Fibonacci" numbers defined on negative indices by

$$F_{n-2} = F_n - F_{n-1}$$
$$F_{-n} = (-1)^{n+1} F_n,$$

and all integers can be represented uniquely by a sum of nega-Fibonacci numbers, taking 0 to be the empty sum.

One question that arises fairly naturally is that of distributivity of circle multiplication over the usual addition. This is tantalizing due to the resulting semiring structure of $(\mathbb{N}, +, \circ)$ this would imply. However, this would mean that

$$m \circ n = (\underbrace{1 + \cdots + 1}_{m}) \circ (\underbrace{1 + \cdots + 1}_{n}) = (1 \circ 1)mn = 3mn$$

with concatenation representing the usual multiplication. The fact that $5 \circ 13 = F_5 \circ F_7 = F_{12} = 144 \neq 3(5)13 = 143$ shows it not to be the case (though distributivity can be shown to hold over clean addition). Knuth actually showed that $m \circ n \sim \sqrt{5}mn$ for large m and n. He also mentioned that Porta and Stolarsky found that $m * n \sim 3.62mn$.

References

[1] D. Knuth, *Fibonacci Multiplication*, App. Math. Lett. Vol.1 pp. 57-60, 1988.

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[4] E. Zeckendorf, *Représentation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas*, Bull. Soc. Roy. Sci. Liège, 1972.