# Riemann Integration

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**Abstract**

This talk is intended to explore the side of Riemann integration which is generally glossed over in calculus and analysis courses. The talk will begin with an overview of some of the historical events which lead up to the invention of integration. Once this has been done, the Riemann integral itself will be defined, and some interesting counterintuitive examples will be discussed.

First of all, what do we mean when we refer to *area*? There are a couple of properties which we might like any sensible notion of area to satisfy:

a) The area of any rectangle should be equal to the product of the lengths of its sides.

b) The area function should be additive: $A(S \cup T) = A(S) + A(T) - A(S \cap T)$.

c) The area function should be invariant under rotation, reflection, and translation.

(If we're being technical, area is a function whose domain is some set $M \subset \mathcal{P}(\mathbb{R}^2)$ and whose range is $\mathbb{R}_{\geq 0}$. The exact nature of $M$ (i.e., the classification of measurable sets in the plane) is beyond the scope of this talk.) From the given axioms, we can derive the standard formula for the area of a right triangle, and from there the area of any triangle. In fact, we are already in the position to derive a nontrivial result which was first given by Archimedes (3rd Century BC) in *The Quadrature of the Parabola*. Written as a letter, it gave a formula for the area enclosed by a parabola and a line segment. An outline of his proof follows.

**Theorem 1:** The area of the region $S$ enclosed by a parabola $P$ and a line $L$ which intersects the parabola is equal to four-thirds of the area of the triangle $T$ whose vertices are given by:

- The two points of intersection of the chord and the parabola.
- The point $Q$ on the parabola such that the tangent to the parabola at that point is parallel to the chord.

**Lemma 1:** The line which is parallel to the axis of symmetry and passes through $Q$ divides the chord into equal segments.

*Proof.* This property was well know in ancient times (Archimedes cited Euclid's *Elements*, rather than prove the result himself), and the proof is easy, if a little computation heavy. It is left as an exercise for the reader. □

*Proof of the Theorem.* We intend to proceed by the method of exhaustion. Notice that the two sides of the triangle other than the original chord are also chords themselves, so we can draw two new triangles in the style of the first one. We have that

- The width of these will be half that of the original triangle.
- The height of these will be a quarter of that of the original triangle.

The first of these follows from Lemma 1. The second can be proven by the methods of geometry, and is relatively straightforward, but messy. So that we don't get bogged down with uninteresting computations, it is also left as an exercise for the reader.

It then follows that each triangle has an eighth of the area of the original triangle. Therefore, if the area of the triangle is denoted by $A(T)$, the area of the two new triangles added will be $\frac{1}{4}A(T)$. If we keep iterating this process, letting $A(S)$ denote the area of the region $S$, we get the relation

$$\begin{aligned} A(S) &= A(T) + \frac{1}{4}A(T) + \frac{1}{16}A(T) + \cdots \\ &= A(T)\left(1 + \frac{1}{4} + \frac{1}{16} + \cdots\right) \\ &= A(T)\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n \\ &= \frac{4}{3}A(T), \end{aligned}$$

as desired. □

Notice that we can use this to find the area between the parabola $y = x^2$ and the line connecting the origin and the point $(a, 0)$. If we decide (for whatever reason) to denote this quantity by

$$\int_0^a x^2 \, \mathrm{d}x,$$

and denote the triangle whose vertices are $(0, 0)$, $(-a, a^2)$, and $(a, a^2)$ by $T$, then we have that

$$\int_0^a x^2 \, \mathrm{d}x = a(a^2) - \frac{1}{2}\left(\frac{4}{3}A(T)\right).$$

Notice that

$$\begin{aligned} A(T) &= \frac{1}{2}a^2(2a) \\ &= a^3, \end{aligned}$$

so

$$\begin{aligned} \int_0^a x^2 \, \mathrm{d}x &= a^3 - \frac{1}{2}\left(\frac{4}{3}a^3\right) \\ &= a^3 - \frac{2}{3}a^3 \\ &= \frac{1}{3}a^3 \end{aligned}$$

I shouldn't have to work too hard to convince you that statements of this form are of interest. Is there a way to make the trick we performed with triangles, or something like it, apply to a more general class of sets in the plane? With the parabola, we took the limit of a sequence of triangles. In general, if we are dealing with regions which are between the graph of some "nice" function and the $x$-axis, it is more convenient to take the limit of a sequence of rectangles.

**Definition:** Let $a < b$; a *partition* of the interval $[a, b]$ is a finite set $a = t_0 < t_1 < \cdots < t_n = b$.

**Definition:** Let $f$ be bounded on $[a, b]$, let $P = \{t_0, \cdots, t_n\}$ be a partition of $[a, b]$, and let

$$m_i = \inf\{f(x) : t_{i-1} \leq x \leq t_i\}$$

Then the *lower sum* of $f$ for $P$ is defined as

$$L(f, P) := \sum_{i=1}^{n} m_i(t_i - t_{i-1}).$$

We might be tempted at this point to let

$$\sup\{L(f, P) : P \text{ a partition of } [a, b]\}$$

be our definition of area for the cases we are discussing. Recall that for the parabola, though, we took what were essentially lower sums, but didn't really justify that the limit that these approached actual was what we wanted (i.e., maybe there was some space inside that was never filled in, even asymptotically). However, we can see that $L(f, P)$, from the way we defined it, will be at least no larger than the value that we want for the area. Therefore, we will define are "upper sum" in a similar fashion to serve as an upper bound.

**Definition:** Let $f$ be bounded on $[a, b]$, let $P = \{t_0, \cdots, t_n\}$ be a partition of $[a, b]$, and let

$$M_i = \sup\{f(x) : t_{i-1} \leq x \leq t_i\}$$

Then the *upper sum* of $f$ for $P$ is defined as

$$U(f, P) := \sum_{i=1}^{n} M_i(t_i - t_{i-1}).$$

When

$$\sup\{L(f, P) : P \text{ a partition of } [a, b]\} = \inf\{U(f, P) : P \text{ a partition of } [a, b]\},$$

we let this be the area of the region between $a$ and $b$ which is bounded above by $f$ and below by the $x$-axis. This particular notion of area called the Riemann integral. We intend to finish up by talking about an incredibly powerful theorem and surprising theorem (which you are surely familiar with), that can be proven from the tools of differential calculus, as well as the small amount of integral calculus presented in this talk.

**Theorem 2** (Fundamental Theorem of Calculus)**:** If $f$ is integrable on $[a, b]$ and $f = g'$ for some function $g$, then

$$\int_a^b f = g(b) - g(a)$$

*Proof.* Let $P = \{t_0, \cdots, t_n\}$ be a partition of $[a, b]$. By the MVT, for all $0 < i \leq n$, there exists $x_i \in [t_{i-1}, t_i]$ such that

$$g(t_i) - g(t_{i-1}) = f(x_i)(t_i - t_{i-1}).$$

If, just like before, we let

$$m_i = \inf\{f(x) : t_{i-1} \le x \le t_i\}$$
$$M_i = \sup\{f(x) : t_{i-1} \le x \le t_i\},$$

then

$$m_i(t_i - t_{i-1}) \le f(x_i)(t_i - t_{i-1}) \le M_i(t_i - t_{i-1})$$
$$m_i(t_i - t_{i-1}) \le g(t_i) - g(t_{i-1}) \le M_i(t_i - t_{i-1})$$
$$\sum_{i=1}^{n} m_i(t_i - t_{i-1}) \le g(b) - g(a) \le \sum_{i=1}^{n} M_i(t_i - t_{i-1})$$
$$L(f, P) \le g(b) - g(a) \le U(f, P).$$

This applies to all partitions, so we must have that

$$\sup\{L(f, P) : P \text{ a partition of } [a, b]\} \le g(b) - g(a) \le \inf\{U(f, P) : P \text{ a partition of } [a, b]\}.$$

But $f$ is integrable on $[a, b]$, so the result follows. □

Math is about find answers to questions, but it is also about finding the right questions to ask in the first place. In general, when you encounter a theorem which you haven't seen before, it is good practice to ask yourself about each part of the hypothesis: Why is this here? What happens if we remove it? If we do that here, we get two questions:

a) If $f$ is integrable on $[a, b]$, does it follow that $f = g'$ for some function $g$?

b) If $f = g'$ for some function $g$, does it follow that $f$ is integrable on $[a, b]$?

This first question is easy. Any (linear) function with a single removable discontinuity is clearly integrable, and derivative functions cannot have removable discontinuities. (n.B. Even worse examples can be constructed, since it can be shown that all monotonic functions are integrable, and monotonic functions with arbitrary countable sets of points of discontinuity can be constructed.)

For the second question, consider the function

$$f(x) = \begin{cases} 2x \sin \dfrac{1}{x^2} - \dfrac{2}{x} \cos \dfrac{1}{x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

on the interval $[-1, 1]$. Notice that if

$$g(x) = \begin{cases} x^2 \sin \dfrac{1}{x^2} & x \neq 0 \\ 0 & x = 0, \end{cases}$$

then $g$ is a primitive for $f$. But $f$ is unbounded on $[-1, 1]$, and therefore is not (Riemann) integrable there.

## References

[1] Michael Spivak, *Calculus*, 4th edition, 2008.

[2] Bernard R. Gelbaum, *Counterexamples in Analysis*, 1964.