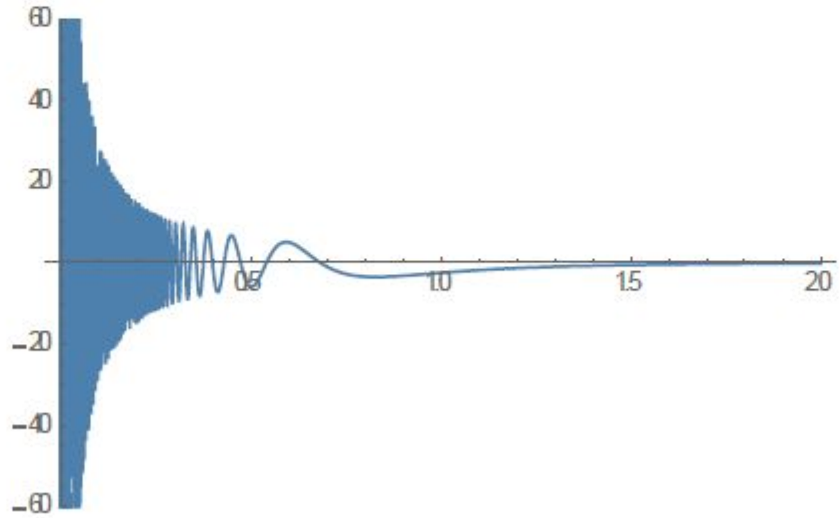
What is... Denjoy Integral?



By Boming Jia

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Handout 1

Georg Friedrich Bernhard Riemann [ˈʁiːman] (listen) (September 17, 1826 – July 20, 1866) was an influential German mathematician who made lasting contributions to analysis, number theory, and differential geometry, some of them enabling the later development of general relativity.

Henri Léon Lebesgue ForMemRS[1] (French: [ɑ̃ʁi leɔ̃ ləbɛɡ]; June 28, 1875 – July 26, 1941) was a French mathematician most famous for his theory of integration, which was a generalization of the 17th century concept of integration—summing the area between an axis and the curve of a function defined for that axis. His theory was published originally in his dissertation *Intégrale, longueur, aire* ("Integral, length, area") at the University of Nancy during 1902.[3][4]

Oskar Perron (7 May 1880 – 22 February 1975) was a German mathematician.

He was a professor at the University of Heidelberg from 1914 to 1922 and at the University of Munich from 1922 to 1951. He made numerous contributions to differential equations and partial differential equations, including the Perron method to solve the Dirichlet problem for elliptic partial differential equations. He wrote an encyclopedic book on continued fractions *Die Lehre von den Kettenbrüchen*.

Arnaud Denjoy (French: [dɑ̃ˈʒwa]; 1884–1974) was a French mathematician.

Denjoy was born in Auch, Gers. His contributions include work in harmonic analysis and differential equations. His integral was the first to be able to integrate allderivatives. Among his students is Gustave Choquet. He is also known for the more general broad Denjoy integral, or Khinchin integral.

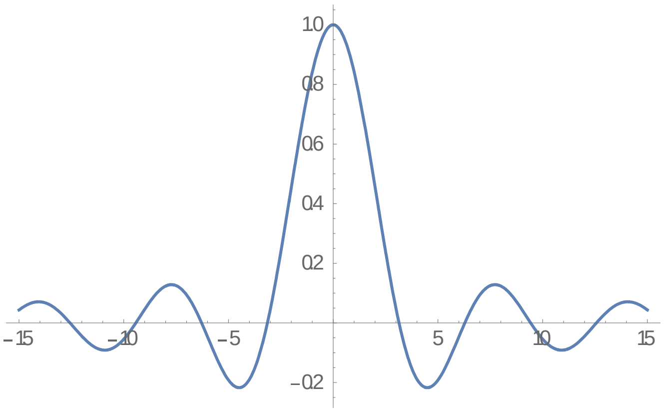
Denjoy died in Paris in 1974.

Ralph Henstock (2 June 1923 – 17 January 2007) was an English mathematician and author. As an Integration theorist, he is notable for Henstock–Kurzweil integral. Henstock brought the theory to a highly developed stage without ever having encountered Jaroslav Kurzweil's 1957 paper on the subject.

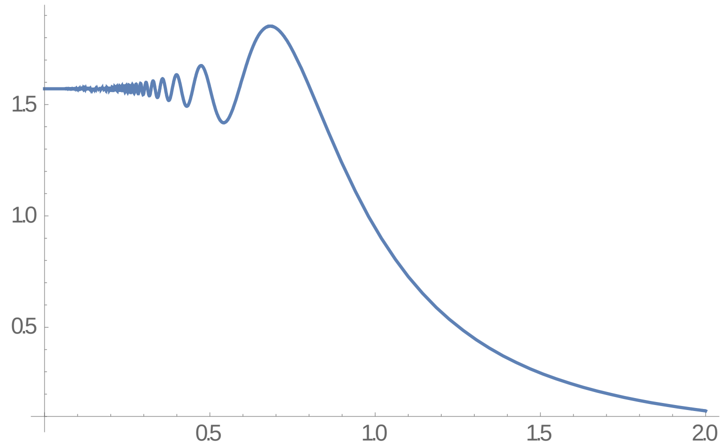
Jaroslav Kurzweil (born 1926) (Czech pronunciation: [ˈjarɔslaf ˈkurtsvajl]) is a Czech mathematician. He is a specialist in ordinary differential equations and defined the Henstock–Kurzweil integral in terms of Riemann sums. Kurzweil has been awarded the highest possible scientific prize of the Czech Republic, the "Czech Mind" of the year 2006, as an acknowledgement of his life achievements.[1][2]

Handout 2

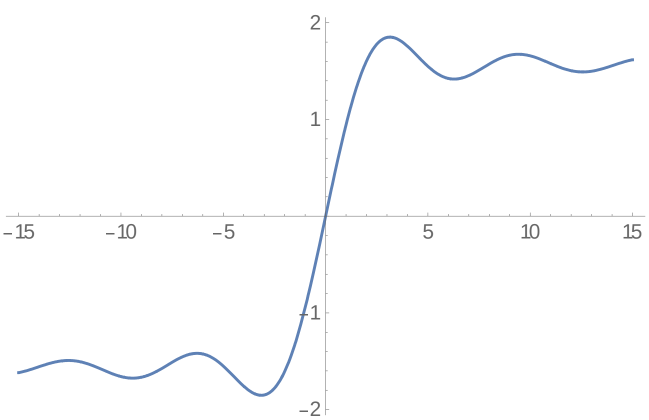
Plot [Sin[t]/t,{t,-15,15]



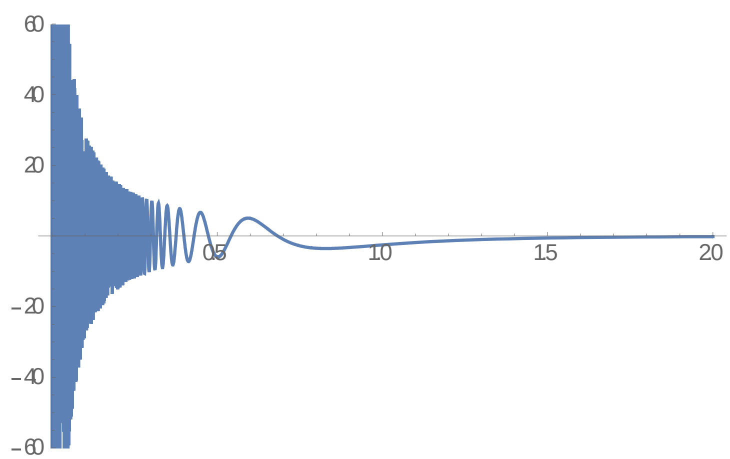
Plot [SinIntegral[1/x^3],{x,0,2}]



Plot [SinIntegral[x],{x,-15,15}]



Plot [-(3/x Sin[1/x^3]),{x,0,2]



Denjoy Integral

Denjoy's Headache

Around 1912, French Mathematician Arnaud Denjoy

Considered integrals such as

$$\int_0^1 \frac{1}{x} \sin \frac{1}{x^3} \, dx$$

To do this, we let

$$f(x) = \begin{cases} \int_0^{\frac{1}{x^3}} \frac{\sin t}{t} \, dt & , x > 0 \\ \frac{\pi}{2} & , x = 0 \end{cases}$$

Then

$$f'(x) = \begin{cases} -\frac{3}{x} \sin \frac{1}{x^3} & , x > 0 \\ 0 & , x = 0 \end{cases}$$

(one-sided derivative)

Thus

$$\int_0^1 \frac{1}{x} \sin \frac{1}{x^3} \, dx = -\frac{1}{3} \int_0^1 f'(x) \, dx$$

$$= -\frac{1}{3} \left(f(1) - f(0) \right)$$

$$= -\frac{1}{3} \left(\int_0^1 \frac{\sin t}{t} \, dt - \frac{\pi}{2} \right)$$

$$= \frac{\pi}{6} - \frac{1}{3} \int_0^1 \frac{\sin t}{t} \, dt \approx 0.208$$

What is the definition of integral?

Riemann	(1854)	
Lebesgue	(1901) (1902)	book (1904)
Denjoy	(1912) two papers eight pages ~~(1915-1917)~~	
	(1915-1917) four papers, ≈ 400 pages	
Perron	(1915)	
Henstock	(1955)	
Kurzweil	(1957)	

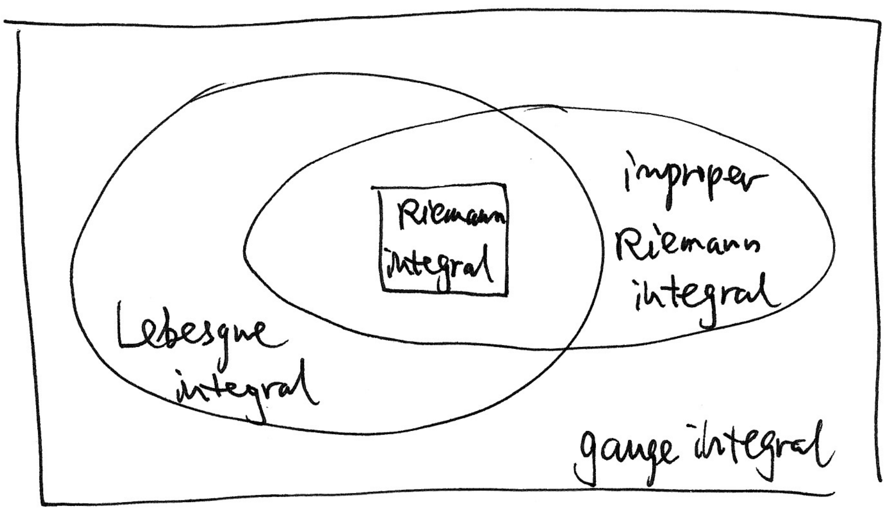
equivalence 1921-1925

indep. work equivalence

$\Updownarrow$ much simpler

"Gauge Integrals"

"Generalized Riemann Integral"



Definitions

① Partition

If $I = [a,b]$ is a non-degenerate compact interval in $\mathbb{R}$, then a partition of I is a finite collection

$$\mathcal{P} := \{ I_i : i = 1, \cdots, n \} = \{ I_i \}_{i=1}^{n}$$

of nonoverlapping compact subintervals I_i, such that

$$I = I_1 \cup \cdots \cup I_n .$$

② Tagged Partition

A tagged partition of interval I is a set of ordered pairs $\dot{\mathcal{P}} := \{ (I_i, t_i) : i = 1, \cdots, n \}$ s.t. $\{I_i\}_{i=1}^{n}$ forms a partition of I, and $t_i \in I_i$ for all i.

③ Riemann Sum of f corresponding to $\dot{\mathcal{P}}$

$$S(f; \dot{\mathcal{P}}) := \sum_{i=1}^{n} f(t_i)\, l(I_i)$$

where $l(I_i)$ is the length of the ith interval.

i.e. if $I_i = [a_i, b_i]$, then $l(I_i) = b_i - a_i$

(4) A <u>gauge</u> on an interval I is a positive valued function $\delta : I \to \mathbb{R}$, i.e. $\delta(t) > 0$ for all $t \in I$.

(15) A tagged partition $\dot{P}$ is <u>δ-fine</u> if δ is a gauge on I, and $I_i \subseteq [t_i - \delta(t_i), t_i + \delta(t_i)]$ for all $i = 1, 2, \cdots, n$.

(we say $\dot{P}$ is <u>subordinate to δ</u> or write <u>$\dot{P} \ll \delta$</u>)

<u>Riemann Integral</u>

<u>Def</u> A function $f : I \to \mathbb{R}$ is said to be R-integrable if $\exists A \in \mathbb{R}$, $\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$ s.t.

if $\dot{P} := \{(I_i, t_i)\}_{i=1}^{n}$ is any tagged partition of I with $\ell(I_i) \leq \delta_\varepsilon$, then $|S(f; \dot{P}) - A| < \varepsilon$.

<u>Generalize Riemann Integral</u>

<u>Def</u> A function $f : I \to \mathbb{R}$ is said to be <u>generalized R-integrable</u> (R^*-integrable) if $\exists B \in \mathbb{R}$, $\forall \varepsilon > 0$, $\exists$ a gauge δ_ε on I s.t.

if $\dot{P} := \{(I_i, t_i)\}_{i=1}^{n}$ is any δ-fine tagged partition then $|S(f; \dot{P}) - B| < \varepsilon$

Existence of δ-fine partition, and Uniqueness of integral

Cousin's Theorem

If $I = [a,b]$ and δ is a gauge on I,

then there exists a partition of I that is δ-fine.

Uniqueness Theorem

There is at most one number B satisfies the definition of R^*-integral of a function f.

Examples

① All R-integrable functions are R^*-integrable

② $f(x) = \begin{cases} 1, & x \in [0,1] \text{ rational} \\ 0, & x \in [0,1] \text{ irrational} \end{cases}$, $\int_0^1 f = 0$.

(Hint. Let $\mathbb{Q} \cap [0,1] = \{r_k : k \in \mathbb{N}\}$. Choose gauge $\delta_\varepsilon(t) = \begin{cases} \frac{\varepsilon}{2^{k+1}} & \text{if } t = r_k \\ 1 & \text{if } t \text{ is irrational} \end{cases}$

③ If $F : [a,b] \to \mathbb{R}$ is differentiable at every point,

then $f = F'$ is R^*-integrable and

$$\int_a^b f = F(b) - F(a)$$

Hints for FTC

Straddle Lemma

Let $F: I \to \mathbb{R}$ be differentiable at a point $t \in I$.

Given $\varepsilon > 0$, there exists $\delta_\varepsilon(t) > 0$ s.t.

if $u, v \in I$ satisfy $t - \delta_\varepsilon(t) \le u \le t \le v \le t + \delta_\varepsilon(t)$

then $\left| F(v) - F(u) - F'(t)(v-u) \right| \le \varepsilon (v-u)$

(Use definition of derivative, and triangular inequality)

Suppose $\dot{P}$ is δ_ε − fine

Now $\left| F(b) - F(a) - S(f; \dot{P}) \right|$

$$= \left| \sum_{i=1}^{n} \left[F(x_i) - F(x_{i-1}) - f(t_i)(x_i - x_{i-1}) \right] \right|$$

$$\le \sum_{i=1}^{n} \left| F(x_i) - F(x_{i-1}) - f(t_i)(x_i - x_{i-1}) \right|$$

(by Straddle Lemma) $\displaystyle \le \sum_{i=1}^{n} \varepsilon (x_i - x_{i-1}) = \varepsilon (b-a)$

Since $\varepsilon > 0$ is arbitrary, we conclude

f is R^*-integrable with $\int_a^b f = F(b) - F(a)$

(4) All proper Lebsgue - integrable functions are R^*-integrable

<u>Characterization</u> $f : [a,b] \to \mathbb{R}$

f is L-integrable $\Longleftrightarrow$ both f and $|f|$ are R^*-integrable

R^*-integral approach	L-integral approach
define R^*-integral	σ-field, measure
check properties	measurable function
consider step functions	indicator functions
$S(x) = \sum c_i \mathbb{1}_{I_i}$	simple functions $\sum c_i \mathbb{1}_{A_i}$
measurable functions	approximate measurable functions by simple functions
$f(x) = \lim\limits_{k\to\infty} S_k(x)$ a.e. on I	Prove Convergence Theorems
$E \subseteq I$ is called <u>measurable</u> if $\mathbb{1}_E$ is measurable.	(monotone, Fatou's Lemma, Dominated
is called <u>integrable</u> if $\int_a^b \mathbb{1}_E$ exists	
define $\lvert E\rvert = \int_a^b \mathbb{1}_E$ to be measure of E	

Convergence Theorems

DCT $\Rightarrow \int_{R^*} \mathbb{1}_E = \int_{\mathcal{L}} \mathbb{1}_E$

MCT $\Rightarrow \int_{R^*} f = \lim\limits_{n\to\infty} \int_{R^*} \varphi_n$

$\int f = \lim\limits_{n\to\infty} \int \varphi_n$

Reference

First, for those who want to see more details of the proofs of the statements in this talk, I suggest to read the following paper.

Bartle, R. (n.d.). Return to the Riemann Integral. *The American Mathematical Monthly,* 625-625.

For an introduction text about Gauge Integral, I recommend a book by the same author of the paper above. Here is a citation of the book.

Bartle, R. (2001). *A modern theory of integration*. Providence, R.I.: American Mathematical Society.

Also, if you are really curious about the original definition of Denjoy Integral, or if you want to know the proof of equivalence of all those definitions of denjoy integrals, including the gauge integral we just see, you might be interested in reading the following book, which is a self-contained treatment of the theory of integration.

Gordon, R. (1994). *The integrals of Lebesgue, Denjoy, Perron, and Henstock*. Providence, R.I.: American Mathematical Society.