# What are Kepler's Laws?

**Notes written by**

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**07/17/2014**

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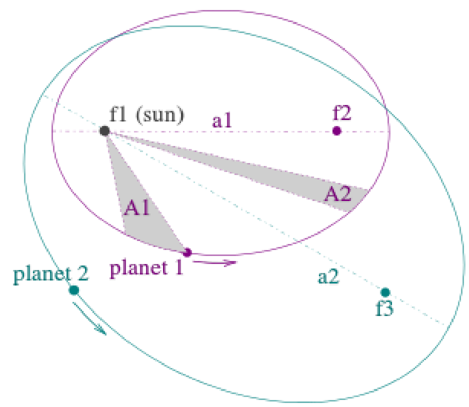
**Kepler's laws of planetary motion** are three scientific laws describing the motion of planets around the Sun. They are

1. The planets move in elliptical orbits, with the Sun at one focus point.
2. The radius vector to a planet sweeps out area at a rate that is independent of its position in the orbits.
3. The square of the period of an orbit $T$, is proportional to the cube of the semi-major-axis length $a$

in equation: $$T^2 = \frac{4\pi^2 a^3}{GM}$$




Here and throughout the note

$G$ stands for the Gravitational Constant which is about $6.67 \times 10^{-11} N \cdot (m/kg)^2$)

$M$ is the mass of the Sun.

We assume the mass of the star is very large compared to the mass of the planets so we can treat the position of the star to be a fixed point in the space. Without this assumption we need to use the idea of reduced mass and modify this equation a little bit.

In 1609 *Johannes Kepler* published his first two laws about planetary motion, by analyzing the astronomical observations of Tycho Brahe. And Kepler's Third Law was published in 1619 (only in the form of proportionality of $T^2$ and $a^3$). In 1687 Isaac Newton showed that his own laws of motion and law of universal gravitation implies Kepler's laws, which is the beginning of all mathematical formulations of Laws of Physics. In this note, I am going sketch the derivation of all three of Kepler's Laws from classical Newtonian mechanics. Also we will see how much information about gravity we can get from Kepler's Laws.

# Newton's laws of physics implies the Kepler's Laws

Newtonian Mechanics Laws tells us essentially the relation $F = ma$ where $F$ is the total external force acted on a point particle with mass $m$ and $a$ is the acceleration of this particle. Since the sizes of the planets and the Sun are very small compared to the distance between the planet and the star, we can treat them as point particle. Another reason that we can treat them as mass points is because they are essentially spherical. And we can show that the Newton's gravity between two solid balls $B_1$ and $B_2$ with mass $M_1$ and $M_2$ whose centers are distance $r$ apart is the same as the gravity between two point mass $M_1$ and $M_2$ with distance $r$.

Newton's Gravity Law describes the gravity between two point mass ($M$ and $m$) with distance $r$ to each other. The relation is $F = \frac{GMm}{r^2}$ where $F$ is the magnitude of the gravity between them. However this is not everything that Newton's Gravity Law says. We know that force, acceleration and position can all be described by vectors. So we can put an arrow above all the vector quantities which has not only a magnitude but also a direction. Thus the Newtonian Mechanics Laws states $\vec{F} = m\vec{a}$ which also tells us the particle accelerates in exactly the direction of the total external force on it. Now we can state the Newton's Gravity Law in the vector form which is as following:

$$\vec{F} = -\frac{GMm}{|\vec{r}|^2}\frac{\vec{r}}{|\vec{r}|}$$

Here $\vec{r}$ is the position vector of $m$ (from $M$) and $\vec{F}$ is the gravity acted on $m$ by $M$. And actually it is also true if we switch "$m$" and "$M$" in the last sentence.

We see this formula is a little more complicated than the formula for magnitude. The minus sign “−” tells us that gravity is always attractive. And the direction of the force is exactly the opposite of $\frac{\vec{r}}{|\vec{r}|}$ which is the direction of the position vector from M to m.

In this note we adopt a useful notation for the time derivative as following:

$$\dot{y} = \frac{dy}{dt} \; and \; \ddot{y} = \frac{d^2y}{dt^2} \text{ for any quantity } y$$

So by definition we have the velocity $\vec{v} = \dot{\vec{r}}$ and the acceleration $\vec{a} = \ddot{\vec{r}}$.

Another piece of important ingredient for the derivation is the angular momentum. For a mass particle $m$ with position vector $\vec{r}$ and velocity $\vec{v}$ (both of which depends on the reference frame), the *angular momentum* of it is a vector $\vec{L}$ defined by $\vec{L} = \vec{r} \times m\vec{v}$ and here “×” is the usual cross product in three dimension. By taking derivative of $\vec{L} = \vec{r} \times m\vec{v}$ and using the relation $\vec{F} = m\vec{a}$ we will get $\dot{\vec{L}} = \vec{r} \times \vec{F}$ where $\vec{F}$ is the gravitational force acted on the planet. Since Newton’s Gravity Theory tells us that $\vec{F}$ is collinear with $\vec{r}$, then we have $\dot{\vec{L}} = 0$, hence $\vec{L}$ is a constant function of time (in physics we say such a quantity is “conserved”). The conservation of angular momentum $\vec{L}$ for a planet tells us two things, one is that $|\vec{L}|$ is a constant with respect to time, the other is the planet always moves in the plane spanned by the velocity of the planet $\vec{v}$ and the position vector $\vec{r}$

If we let $r = |\vec{r}|$ and set up a coordinate system in the plane of the span of $\vec{v}$ and $\vec{r}$ then we can express $\vec{r}$ as $(r\,cos\theta, r\,sin\theta)$ for some $\theta \in [0,2\pi)$. Now we can see that the magnitude of the angular momentum of our planet is $L = \left|\vec{L}\right| = mr^2\dot{\theta}$ which is conserved. Also we have $|\vec{v}|^2 = \dot{r}^2 + (r\dot{\theta})^2$.

We know that the *kinetic energy* for a particle with mass $m$ and velocity $\vec{v}$ is $\frac{1}{2}m|\vec{v}|^2$ (by integrating $\vec{F} = m\vec{a}$ with respect to spatial displacement $d\vec{x}$). Therefore the expression for the kinetic energy is $\frac{1}{2}m\dot{r}^2 + \frac{1}{2}m(r\dot{\theta})^2$. And the *potential energy* for Newton's Gravitational Force is $-\frac{GMm}{r}$ (by integrating the gravity equation $F = \frac{GMm}{r^2}$ as a function of $r$). Let $\alpha = GMm$ (throughout this note we adopt this convention), then the potential energy is $-\frac{\alpha}{r}$. Also if we use E to denote the total energy of the planet, then we have

$$\frac{1}{2}m\dot{r}^2 + \frac{1}{2}m(r\dot{\theta})^2 - \frac{\alpha}{r} = E$$

Since $L = mr^2\dot{\theta}$, we can rewrite the above formula as

$$\frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{\alpha}{r} = E$$

A bit of algebra applied to $L = mr^2\dot{\theta}$ and this formula we can get

$$(\frac{1}{r^2}\frac{\dot{r}}{\dot{\theta}})^2 = -\frac{1}{r^2} + \frac{2m\alpha}{rL^2} + \frac{2mE}{L^2}$$

We know that $\frac{\dot{r}}{\dot{\theta}} = \frac{dr}{d\theta}$ so the above equation can be treated as a differential equation of the function $r(\theta)$. We can solve it by making a series of substitutions: $y = \frac{1}{r}, z = y - \frac{m\alpha}{L^2}, \varepsilon = \sqrt{1 + \frac{2EL^2}{m\alpha^2}}$, then we will get the general solution is $z = \frac{m\alpha}{L^2}\ \varepsilon\ cos(\theta - \theta_0)$ where $\theta_0$ is an arbitrary constant. We can take care of the axis such that $\theta_0 = 0$, hence we get $z = \frac{m\alpha}{L^2}\ \varepsilon\ cos\,\theta$ and in terms of $r$ and $\theta$ we have $r = \frac{L^2}{m\alpha}\ \frac{1}{1-\varepsilon\ cos\theta}$. Let $k = \frac{L^2}{m\alpha}$, then we have

$$r = \frac{k}{1 - \varepsilon\ cos\theta} \quad where\ k = \frac{L^2}{m\alpha}\ and\ \varepsilon = \sqrt{1 + \frac{2EL^2}{m\alpha^2}} \qquad (1)$$

This equation describes the trajectory of the planet, and if we assume $\varepsilon < 1$ then calculate the equation in Cartesian coordinate by substitute $x = r\,cos\theta$ and $y = r\,sin\theta$, with some algebra we can get

$$\frac{(x+c)^2}{a^2} + \frac{y^2}{b^2} = 0$$

$$where\ a = \frac{k}{1-\varepsilon^2}, b = \frac{k}{\sqrt{1-\varepsilon^2}}\ and\ c = \frac{k\varepsilon}{1-\varepsilon^2}$$

With some analytic geometry, this equation says that the trajectory is an ellipse with one focus at the origin. Hence we have demonstrated Kepler's First Law. (Question: what will happen if $\varepsilon = 1$ or $\varepsilon > 1$ ? In what cases these will happen?)

Now we are going to prove Kepler's Second Law. First let $A(t)$ be the area that the radius vector of the planet sweeps in the time interval 0 to $t$, then the claim of the second law is just $A'(t)$ is constant. The proof is as following:

$$A'(t) = A'(\theta)\dot{\theta} = \frac{r^2\dot{\theta}}{2} = \frac{L}{2m}$$

And $\frac{L}{2m}$ is a constant. The second equality holds because the shape that $\vec{r}$ sweeps in a very small angle $d\theta$ is approximately a triangle with area $\frac{1}{2}r^2 d\theta$.

Now we demonstrate Kepler's Third Law. Let $T$ be the period of the orbit, i.e. the time it takes the planet to finish a full cycle of the orbit. And for simplicity here we let $A = A(T)$. Since the rate between sweeping area and time is the constant $\frac{L}{2m}$, then we have $A = \frac{LT}{2m}$. Also from the property of the ellipse we know that $A = \pi ab$ where $= \frac{k}{1-\varepsilon^2}, b = \frac{k}{\sqrt{1-\varepsilon^2}}$ , $k = \frac{L^2}{m\alpha}$ and $\alpha = GMm$. With a bit algebra we get A, L, b, k, $\alpha$ all cancelled and end up with

$$T^2 = \frac{4\pi^2 a^3}{GM}$$

Which is exactly what Kepler's Third Law states.

# Kepler's Laws indicate Newton's Gravity Law

First we state Kepler's laws in a more mathematical way.

**First Law:** $$r = k - \epsilon\, x \tag{1}$$

where $k, \epsilon$ are constants, $r$ is the length of the radius vector and $x$ is the $x$-coordinate of the planet in the Cartesian coordinate with the Sun at the origin and major axis as $x$-axis.

**Second Law:** $$\int_0^x y\, dx - \frac{xy}{2} = D(t - t_0) \tag{2}$$

where D is a constant, y is the y-coordinate of the planet in the Cartesian coordinate and $t_0$ will be different constants whenever the planet crosses the x-axis. (Draw a picture, then you will understand this better)

**Third Law:** $$\frac{D^2}{k} = H \tag{3}$$

where $H$ is a constant independent of time

It might be a little bit hard to see why the third law is in this form. But if you imitate the proof of the third law from Newtonian Mechanics in the last session, you can show that

$$T^2 = \frac{\pi^2 a^3 k}{D^2}$$

Then it is clear that indeed the Third Law tells us that $\frac{D^2}{k}$ is a constant.

Now we are going to show that these three laws tell us the acceleration is proportional to the inverse square of the radius, which is the most essential part of Newton's Law of Gravity.

Differentiate (1) with respect to time, we get $\dot{r} = -\epsilon \dot{x}$. Differentiate the relation $r^2 = x^2 + y^2$ we have $r\dot{r} = x\dot{x} + y\dot{y}$. Thus

$$\frac{1}{r}(x\dot{x} + y\dot{y}) = -\epsilon \dot{x} \tag{4}$$

Differentiate (2) with respect to time we get

$$y\dot{x} - x\dot{y} = 2D \tag{5}$$

Solve for $\dot{x}$ from (1), (4) and (5), we will get

$$\dot{x} = \frac{2D}{k}\frac{y}{r} \tag{6}$$

We know that $\frac{d}{dt}\left(\frac{y}{r}\right) = \frac{\dot{y}}{r} - \frac{y\,\dot{r}}{r^2} = \frac{r^2\dot{y} - y(\mathrm{x\dot{x}+y\dot{y}})}{r^3} = \frac{x^2\dot{y} - xy\dot{\mathrm{x}}}{r^3} = -\frac{x}{r^3}(y\dot{x} - x\dot{y})$, then we have $\ddot{x} = -\frac{2D}{k}\frac{x}{r^3}(y\dot{x} - x\dot{y})$. Then by (5) we get

$$\ddot{x} = -\frac{4D^2}{k}\frac{x}{r^3}$$

Differentiate (5) with respect to time, we have $y\ddot{x} - x\ddot{y} = 0$. Together with the above equation we get

$$\ddot{y} = -\frac{4D^2}{k}\frac{y}{r^3}$$

Therefore in the vector form

$$\ddot{\vec{r}} = -\frac{4D^2}{k}\frac{\vec{r}}{|\vec{r}|^3}$$

According to the formula $\vec{F} = m\vec{a}$ and the definition $\vec{a} = \ddot{\vec{r}}$ we have that

$$\vec{F} = -\frac{4mH}{|\vec{r}|^2}\frac{\vec{r}}{|\vec{r}|}$$

This formula contains the most essential part of Newton's Law of Gravity, i.e. the magnitude of the gravity is inversely proportional to the square of the distance.

Therefore the Kepler's Laws are almost equivalent to the Newton's Law of Gravity. Nowadays people know that these laws are all approximately correct, and we have a better theory for gravity, which is Einstein's well-known General Theory of Relativity. And the Kepler's Laws correspond to the General Relativity are left to readers to investigate.

# Reference

1. Hyman, A. T. (1993). *The Mathematical Relationship between Kepler's Laws and Newton's Laws,* The American Mathematical Monthly, Vol. 100, No. 10 (Dec., 1993), pp. 932-936

2. Morin, D. J. (2008). *Introduction to classical mechanics: with problems and solutions*. Cambridge, UK: Cambridge University Press.

3. Wikipedia (2014, July). *Kepler's laws of planetary motion.* http://en.wikipedia.org/wiki/Kepler%27s_laws_of_planetary_motion

# Recommended Readings

1. Goodstein, D. L., & Goodstein, J. R. (1996). *Feynman's lost lecture: the motion of planets around the sun*. New York: Norton.

This book provides an elementary (without using differential equations) derivation of Kepler's Laws from Newtonian mechanics.

2. Tao, T. (2007). *The Cosmic Distance* Ladderhttp://www.math.ucla.edu/~tao/preprints/Slides/Cosmic%20Distance%20Ladder.ppt

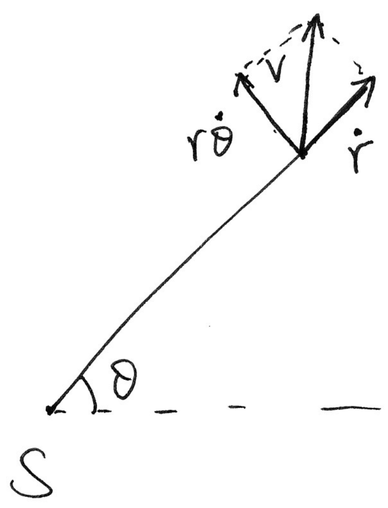
This PowerPoint slideshow is for the talk The Cosmic Distance that was given by Terence Tao at the 2010 Einstein Lecture held by AMS. This talk overviews historically how human beings get the distance or size of different Astronomical Objects. If you are interested in really understanding how people get convinced sun is bigger than the Earth and is further than the Moon, then this talk is for you. Here is the link for the website for the talk:

http://www.ams.org/meetings/lectures/einstein-2010

# Attachments

The following pages are the hand-written drafts and handouts of the talk, which contains more details of algebraic calculations that are omitted above and some basic knowledge in physics and analytic geometry.

$r := |\vec{r}|$ $\qquad \dot{r} := \frac{dr}{dt}, \quad \dot{\theta} := \frac{d\theta}{dt}$, etc.




$$\frac{1}{2} m \dot{r}^2 + \frac{1}{2} m (r\dot{\theta})^2 - \frac{\alpha}{r} = E$$

Let $\vec{L} := \vec{r} \times (m\vec{v})$

then
$$\dot{\vec{L}} = \dot{\vec{r}} \times (m\vec{v}) + \vec{r} \times (m\dot{\vec{v}})$$
$$= \underbrace{\vec{v} \times (m\vec{v})}_{0} + \vec{r} \times (m\vec{a})$$
$$= \vec{r} \times \vec{F} =: \vec{\tau}$$

$$F(r) = -\frac{GMm}{R^2} \cdot \frac{\vec{r}}{R} = -\frac{\alpha}{R^2} \cdot \frac{\vec{r}}{r}$$
$$V(r) = -\frac{\alpha}{R}$$

$$\vec{F} \parallel \vec{r} \implies \dot{\vec{L}} = 0 \implies \vec{L} = \text{const}.$$

Let $L = |\vec{L}| = r(mr\dot{\theta}) = mr^2\dot{\theta}$

$$\frac{1}{2} m \dot{r}^2 + \frac{L^2}{2mr^2} - \frac{\alpha}{r} = E$$

$\boxed{\times 2m}$
$$m^2 \dot{r}^2 = -\frac{L^2}{r^2} + \frac{2m\alpha}{r} + 2mE$$

$\boxed{\times \frac{1}{(mr^2\dot{\theta})^2} = \frac{1}{L^2}}$
$$\left(\frac{1}{r^2} \frac{\dot{r}}{\dot{\theta}}\right)^2 = -\frac{1}{r^2} + \frac{2m\alpha}{rL^2} + \frac{2mE}{L^2}$$

Let $y = \frac{1}{r}$, then $\frac{dy}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{\dot{r}}{\dot{\theta}}$

thus
$$\left(\frac{dy}{d\theta}\right)^2 = -y^2 + \frac{2m\alpha}{L^2} y + \frac{2mE}{L^2}$$
$$= -\left(y - \frac{m\alpha}{L^2}\right)^2 + \left(\frac{m\alpha}{L^2}\right)^2 + \frac{2mE}{L^2}$$

Let $z = y - \dfrac{m\alpha}{L^2}$, then

$$\left(\frac{dz}{d\theta}\right)^2 = -z^2 + \underbrace{\left(\frac{m\alpha}{L^2}\right)^2 \overbrace{\left(1 + \frac{2EL^2}{m\alpha^2}\right)}^{\varepsilon^2}}_{= B^2}$$

$z = B\cos(\theta - \theta_0)$ is the general solution.

Pick axis such that $\theta_0 = 0$

then $$\frac{1}{r} - \frac{m\alpha}{L^2} = \frac{m\alpha}{L^2}\,\varepsilon\cos\theta$$

$$\Rightarrow \quad \frac{1}{r} = \frac{m\alpha}{L^2}(1 + \varepsilon\cos\theta)$$

$$\Rightarrow \quad r = \frac{L^2}{m\alpha} \cdot \frac{1}{1 - \varepsilon\cos\theta}$$

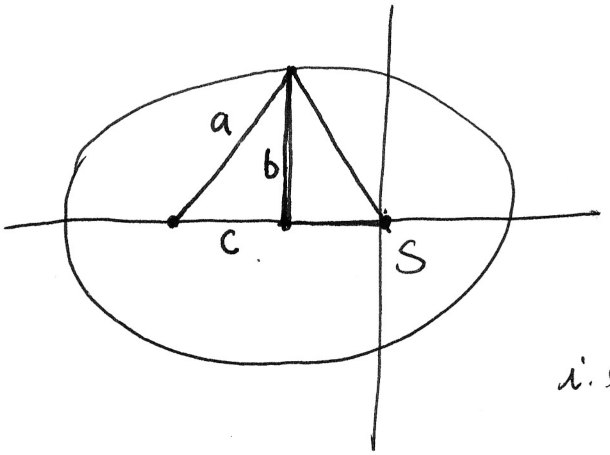
Let $k = \dfrac{L^2}{m\alpha}$, then $r = \dfrac{k}{1 - \varepsilon\cos\theta}$

$$\Rightarrow r + \varepsilon\underbrace{r\cos\theta}_{x} = k \quad \Rightarrow r = k - \varepsilon x \qquad (1)$$

$\Rightarrow$ ... (hand out)

$$\Rightarrow \frac{\left(x + \frac{k\varepsilon}{1-\varepsilon^2}\right)^2}{a^2} + \frac{y^2}{b^2} = 0$$

where $a = \dfrac{k}{1-\varepsilon^2}$, $b = \dfrac{k}{\sqrt{1-\varepsilon^2}}$

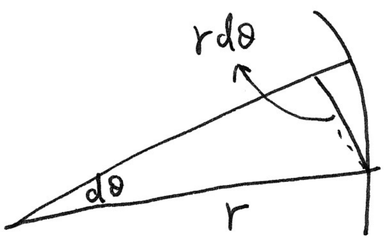



S is a focus

iff $k\left(\frac{1}{1-\varepsilon} - \frac{1}{1+\varepsilon}\right) = 2c$

i.e. $\frac{k\varepsilon}{1-\varepsilon^2} = c$

We know $c = \sqrt{a^2-b^2} = \sqrt{\left(\frac{k}{1-\varepsilon^2}\right)^2 - \left(\frac{k}{\sqrt{1-\varepsilon^2}}\right)^2} = \frac{k\varepsilon}{1-\varepsilon^2}$. $\square$




$$\frac{dA}{dt} = \frac{dA}{d\theta}\cdot\frac{d\theta}{dt} = \frac{r^2}{2}\cancel{\frac{d\theta}{dt}}\,\dot{\theta} = \frac{L}{2m} = \text{const}.$$

$\uparrow$ by $dA = \frac{1}{2}\cdot r\cdot r\,d\theta$

$$A = \frac{LT}{2m}, \qquad A = \pi ab$$

$$\pi^2 a^2 b^2 = \frac{L^2T^2}{4m^2}$$

We know $b = a\sqrt{1-\varepsilon^2} \Rightarrow b^2 = a^2(1-\varepsilon^2)$

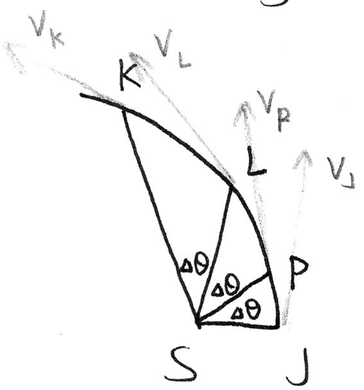
thus $\pi^2 a^4 = \left(\frac{L^2}{m(1-\varepsilon^2)}\right)\frac{T^2}{4m}$

We know $k = \frac{L^2}{m\alpha}$, $a = \frac{k}{1-\varepsilon^2} \Rightarrow \alpha a = \frac{L^2}{m(1-\varepsilon^2)}$

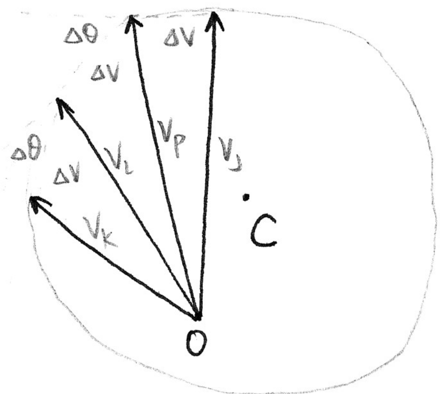
thus $\pi^2 a^4 = \frac{\alpha a T^2}{4m} \Rightarrow T^2 = \frac{4\pi^2 a^3}{GM}$

$(\alpha = GMm)$

trajectory

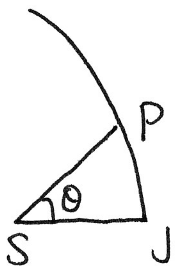


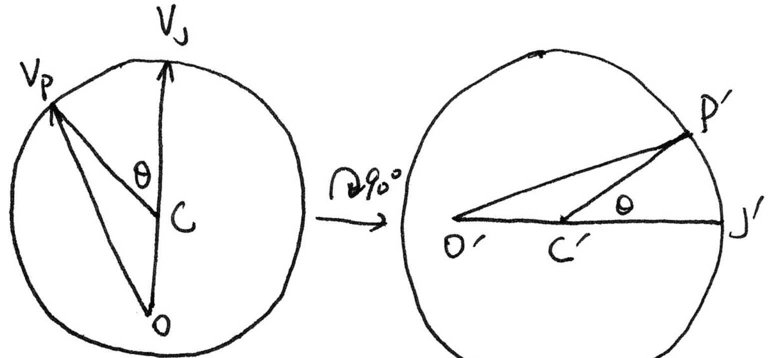

velocity vector $\vec{V_i}$

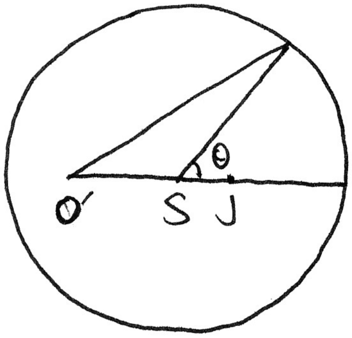


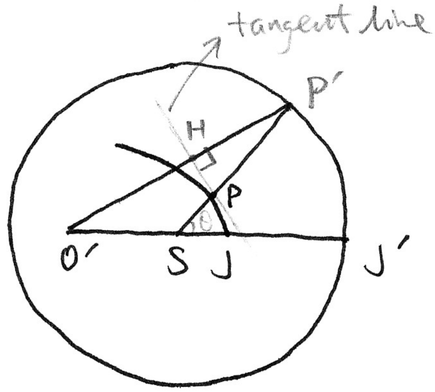

$$\frac{dv}{dt} = \frac{F}{m} \propto \frac{1}{R^2}, \quad \frac{d\theta}{dt} = \frac{d\theta}{dA}\cdot\frac{dA}{dt} \propto \frac{1}{\frac{dA}{d\theta}} \propto \frac{1}{R^2}$$

$$\Rightarrow \frac{dv}{dt} \propto \frac{d\theta}{dt} \quad \Rightarrow \quad \Delta V \propto \Delta\theta \quad \Rightarrow \quad V_i \in \odot C, \ \forall i$$




Given $O'J = JJ'$,

then $PH \perp O'P'$, and $O'H = HP'$ for all $\theta$

Proof. Let $\frac{O'H}{O'P'} = \lambda$, we show $\lambda = \frac{1}{2}$.

Let the radius $SP' = r$, $O'S = c$,

then $\overrightarrow{SO'} = (-c, 0)$, $\overrightarrow{SP'} = (r\cos\theta, r\sin\theta)$

$$\overrightarrow{SH} = \overrightarrow{SO'} + \overrightarrow{O'H} = \overrightarrow{SO'} + \lambda \overrightarrow{O'P'} \cancel{= \overrightarrow{SO'} + \lambda(\overrightarrow{SP'} - \overrightarrow{SO'})}$$

$$= \overrightarrow{SO'} + \lambda(\overrightarrow{SP'} - \overrightarrow{SO'}) = \lambda \overrightarrow{SP'} + (1-\lambda)\overrightarrow{SO'}$$

$$= (\lambda r\cos\theta - (1-\lambda)c, \lambda r\sin\theta)$$

By second Law $\overrightarrow{SP'} \cdot \overrightarrow{O'P'} = \text{const.}$

$$\overrightarrow{SP'} \cdot \overrightarrow{O'P'} = \overrightarrow{O'P'} \cdot \overrightarrow{SH}$$

$$= (r\cos\theta + c, r\sin\theta) \cdot (\lambda r\cos\theta - (1-\lambda)c, \lambda r\sin\theta)$$

$$= \lambda r^2\cos^2\theta + \lambda rc\cos\theta - (1-\lambda)rc\cos\theta - (1-\lambda)c^2 + \lambda^2 r\sin\theta$$

$$= \lambda r^2 + (2\lambda - 1)rc\cos\theta - (1-\lambda)c^2$$

$\theta = 0$, $P \to J$, $P' \to J'$

$$\overrightarrow{SP'} \cdot \overrightarrow{O'P'} = |O'J'| \cdot |SJ| = (r+c)\left(\frac{r+c}{2} - c\right) = \frac{1}{2}(r^2 - c^2)$$

Thus $\lambda r^2 + (2\lambda-1)rc\cos\theta - (1-\lambda)c^2 = \frac{1}{2}(r^2 - c^2)$

$$\Rightarrow (\lambda - \tfrac{1}{2})r^2 + 2(\lambda - \tfrac{1}{2})rc\cos\theta + (\lambda - \tfrac{1}{2})c^2 = 0$$

$$\Rightarrow (\lambda - \tfrac{1}{2})(r^2 + c^2 + 2rc\cos\theta) = 0$$

Since $r^2 + c^2 + 2rc\cos\theta \geq r^2 + c^2 - 2rc = (r-c)^2 > 0$

hence $\lambda - \frac{1}{2} = 0 \Rightarrow \lambda = \frac{1}{2}$

$\square$

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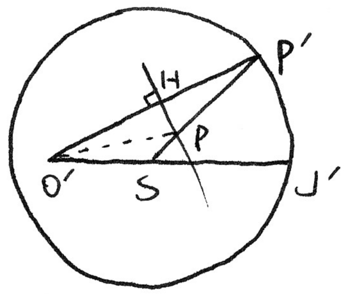



$HP \perp O'P'$, $O'H = P'H$

$\Rightarrow O'P = P'P \Rightarrow O'P + SP = SP = r$

$\Rightarrow P$ lies on an ellipes.

Excercise.

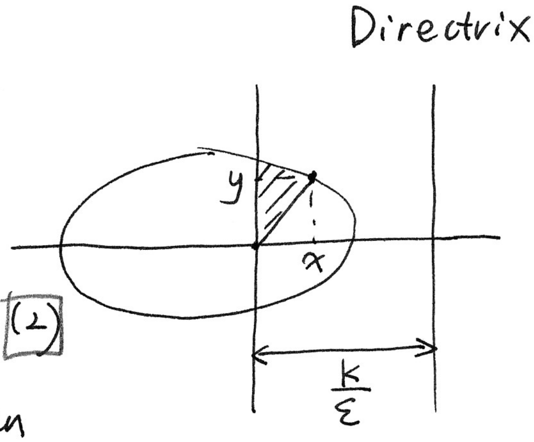
We assumed implicitly $O'$ lies in circle $S$,

what if $O'$ lies on the circle, or outside

the circle.

Show that it matches with the result

we showed before ($\varepsilon = 1$, $\varepsilon > 1$).




$$r = k - \varepsilon x \quad \boxed{(1)}$$

$$\int_0^x y\,dx - \frac{xy}{2} = D(t - t_0) \quad \boxed{(2)}$$

(We need to adjust $t_0$, when the planet crosses $x$-axis)

$$a = \frac{k}{1-\varepsilon^2}, \quad b = \frac{k}{\sqrt{1-\varepsilon^2}}$$

then $\quad A = \pi ab \Rightarrow T = \dfrac{\pi ab}{D}$

$$\Rightarrow T^2 = \frac{\pi^2 a^2 b^2}{D^2} = \frac{\pi^2 a^3 k}{D^2}$$

Third Law said $\quad \dfrac{D^2}{k} = H \quad \boxed{(3)}$

where $H$ is a constant that is independent for all planets.

$(1)' \Rightarrow \quad \dot{r} = -\varepsilon \dot{x}$

since $r^2 = x^2+y^2$, then $r\dot{r} = x\dot{x} + y\dot{y}$

thus $\frac{1}{r}(x\dot{x}+y\dot{y}) = -\varepsilon \dot{x}$ $\quad \boxed{(4)}$

$(2)' \Rightarrow \quad y\dot{x} - x\dot{y} = 2D$ $\quad \boxed{(5)}$

$$\left.\begin{aligned} (4) &\Rightarrow x^2\dot{x} + xy\dot{y} = -r\cdot\varepsilon x\cdot\dot{x} \\ &\overset{(1)}{\Longrightarrow} x^2\dot{x} + r(k-r)\dot{x} + xy\dot{y} = 0 \\ (5) &\Rightarrow y^2\dot{x} - xy\dot{y} = 2Dy \end{aligned}\right\} (+)$$

thus $r^2\dot{x} + r(k-r)\dot{x} = 2Dy$

$\Rightarrow \quad \dot{x} = \frac{2D}{k}\cdot\frac{y}{r}$ $\quad \boxed{(6)}$

$(5)' \Rightarrow \quad y\ddot{x} - x\ddot{y} = 0$ $\quad \boxed{(7)}$

We know $\frac{d}{dt}\left(\frac{y}{r}\right) = \frac{\dot{y}}{r} - \frac{y\dot{r}}{r^2} = \frac{r^2\dot{y}}{r^3} - \frac{y(x\dot{x}+y\dot{y})}{r^3}$

$$= \frac{x^2\dot{y} - xy\dot{x}}{r^3} = \frac{x}{r^3}(x\dot{y}-y\dot{x})$$

$(6)' \Rightarrow \quad \ddot{x} = \frac{2D}{k}\cdot\frac{x}{r^3}(x\dot{y}-y\dot{x}) \overset{\text{by (5)}}{=\!=} -\frac{4D^2}{k}\cdot\frac{x}{r^3}$

$$\overset{\text{by (3)}}{=\!=} -\frac{4H\cdot x}{r^3}$$

$(7) \Rightarrow \quad \ddot{y} = \frac{y}{x}\cdot\ddot{x} = -\frac{4Hy}{r^3}$

Therefore $\ddot{\vec{r}} = -\frac{4H}{r^2} \cdot \frac{\vec{r}}{r}$

So $\vec{F}_g = m\ddot{\vec{r}} = -\frac{4mH}{r^2} \cdot \frac{\vec{r}}{r}$

If we carefully track all the constant, we will see this is consistent with the result we got first time.

If $D = \frac{L}{2m}$, $k = \frac{L^2}{m\alpha}$, $\alpha = GMm$

then $H = \frac{D^2}{k} = \frac{\alpha}{4m} = \frac{GM}{4}$

thus $\vec{F}_g = -\frac{GMm}{r^2} \cdot \frac{\vec{r}}{r}$

Handout for Boming's Talk about "Kepler's Laws"

Convention

$M$ is the mass of the Sun

$m$ is the mass of the planet

$T$ is the time period of the planet

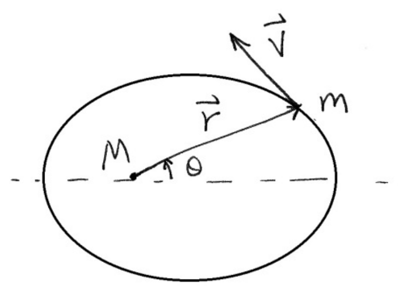
$A$ is the area the planet's orbit enclosing

$G$ is the gravitational constant

$\alpha = GMm$ ≈ 6.67×10^(−11) N·(m/kg)2

$\vec{F}_g$ stands for gravitational force.

$V(r)$ stands for gravitational potential energy.




$\vec{r}$ is the position vector of the planet

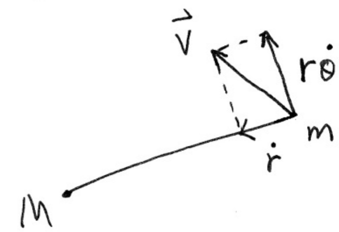
$\vec{v}$ is the velocity vector of the planet

$\theta$ is the angle between the axis (which will be specified) and $\vec{r}$.

$\dot{\vec{r}} := \frac{d\vec{r}}{dt}$, $\ddot{\vec{r}} := \frac{d^2\vec{r}}{dt^2}$, similar for $\dot{x}$, $\ddot{x}$, etc.

Prerequisite in Physics

$r := |\vec{r}|$ then




$\vec{F} = m\vec{a}$ , where $\vec{a} = \ddot{\vec{r}} = \dot{\vec{v}}$

Work $W_F(x_1, x_2) := \int_{x_1}^{x_2} \vec{F}(x) \cdot d\vec{x} = \int_{x_1}^{x_2} m \cdot \frac{d\vec{v}}{dt} \cdot d\vec{x}$

$$= m \int_{v_1}^{v_2} \frac{d\vec{x}}{dt} \cdot d\vec{v} = m \int_{v_1}^{v_2} \vec{v} \cdot d\vec{v} = \frac{1}{2} m \vec{v}^2 \Big|_{v_1}^{v_2}$$

$\frac{1}{2} m \vec{v}^2 \rightarrow$ kinetic energy K.E.

Energy $E$ satisfies $\Delta E = 0$ for all $\Delta t$

and $E = K.E. + P.E.$

So $\Delta P.E. = -\Delta K.E. = -W_F(x_1, x_2)$

Set $P.E._g(\infty) = 0$ , then $P.E._g(r) = W_{F_g}(r, \infty) = \int_r^\infty F_g(r)\, dr$

If $F_g(r) = -\frac{GMm}{r^2}$ , then $P.E._g(r) = -\frac{GMm}{r} = -\frac{\alpha}{r}$

$\alpha = GMm$

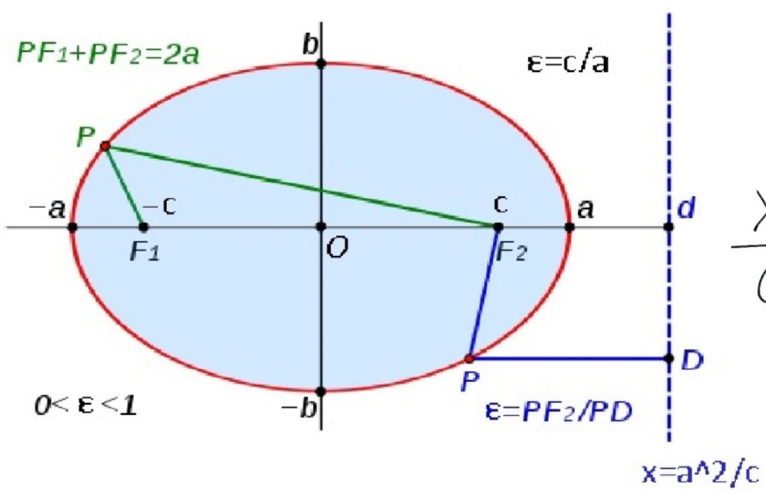
# Kepler's Laws

First Law: The planets move in elliptical orbits, with the sun at one focus

Second Law: The radius vector to a planet sweeps out area at a rate that is independent of its position in the orbit.

Third Law: The square of the period of an orbit, T, is proportional to the cube of the semi-major-axis length a

i.e. $$T^2 = \frac{4\pi^2 a^3}{GM}$$ where a is the following a




$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

This picture is taken and modified from wikipedia "ellipse" page

The following part might be a little bit mysterious until you see the problem appears naturally in the talk.

$$r = \varepsilon x - k \quad \Rightarrow \quad r^2 = \varepsilon^2 x^2 - 2\varepsilon x k + k^2$$

$$\Rightarrow \quad x^2 + y^2 = k^2 - 2k\varepsilon x + \varepsilon^2 x^2$$

$$\Rightarrow \quad (1-\varepsilon^2)\, x^2 + 2k\varepsilon x + y^2 = k^2$$

$$\Rightarrow \quad x^2 + 2 \cdot \frac{k\varepsilon}{1-\varepsilon^2}\, x + \frac{y^2}{1-\varepsilon^2} = \frac{k^2}{1-\varepsilon^2}$$

$$\Rightarrow \left(x + \frac{k\varepsilon}{1-\varepsilon^2}\right)^2 + \frac{y^2}{1-\varepsilon^2} = \frac{k^2}{1-\varepsilon^2} + \frac{k^2\varepsilon^2}{(1-\varepsilon^2)^2} = \frac{k^2}{(1-\varepsilon^2)^2}$$

$$\Rightarrow \quad \frac{\left(x + \frac{k\varepsilon}{1-\varepsilon^2}\right)^2}{\left(\frac{k}{1-\varepsilon^2}\right)^2} + \frac{y^2}{\left(\frac{k}{\sqrt{1-\varepsilon^2}}\right)^2} = 1$$