THE RICCI FLOW

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1. Introduction

Since the turn of the 21st century, the Ricci flow has emerged as one of the most important geometric processes. It has been used to prove several major theorems in differential geometry and topology. In this talk we will try to provide intuition about what it is and how it behaves. One should think of the Ricci flow as being a "heat equation for curvature"[1] and we will try to explain what that means. Instead of proving the various facts about the flow that we assert in the talk, we have included a section at the end to give more technical details and the citations go into more depth still.

2. History

In 1900, Henri Poincaré put for a conjecture that colloquially states that "If it walks like a sphere and it quacks like a sphere, it is a sphere."[2] This statement, known as the Poincaré conjecture, became one of the early questions in a field now called "low-dimensional topology" and proved itself to be extremely subtle and intractable. It attracted the attention of many great mathematicians and attempts to solve it lead to the development of powerful tools. One could devote an entire seminar (or an entire career [9]) to this problem but we mention it in passing due to its relation to the Ricci flow.

In the early 1980s, Richard Hamilton put forth an ambitious program to attack the Poincaré conjecture. He had been studying work by Eells and Sampson [2] that used ideas from heat theory to study harmonic maps. He thought a similar approach might be able to finally prove the question first posed by Poincaré. He proposed a process called the Ricci flow that would deform the shape of a space and hopefully allow its curvature to dissipate

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[1]This is technically not correct but is far too valuable an intuition to discard because of technical details.

[2] More formally, the conjecture is that the three-dimensional sphere is the only simply connected compact three-dimensional manifold.

throughout the space. If the space satisfied the hypotheses of the Poincaré conjecture, he hoped the space would evolve into a round sphere and that he could prove the conjecture with this nicer space. He was able to make this approach work in two dimensions[3] as a proof of concept, but he found that in three dimensions the flow would sometimes violently rip apart spaces and so was unable to finish the proof.

The rest of the story is already a part of mathematical lore. In 2000, the Poincaré conjecture was named as one of the Clay Millenium Prize problems and earned a \$1,000,000 bounty for its solution. No one expected it to be solved in the near future. In 2002, a terse preprint appeared on the arxiv containing major breakthroughs in the Ricci flow. Over the next year, two more preprints followed the first claiming to prove not only the Poincaré conjecture, but a large generalization of it known as Thurston's Geometrization Conjecture. The author, Grigori Perelman was an eccentric genius who had toiled in isolation for nearly a decade crafting his proof. Because of the brevity of the papers and Perelman's reclusive nature, it took several years and an authorship scandal for the mathematical community to accept the work as being correct and due to Perelman. Nonetheless, in 2006, Perelman was awarded a Fields medal for his efforts. He declined the award and later turned down the Millenium Prize, as well. He has since withdrawn from mathematics and lives with his mother in St. Petersburg, Russia.

Nonetheless, it is hard to think of praise too high for Perelman's work. It is one of the greatest mathematical works of all time and a marvel of the power of hard analysis. It is also the main reason why the Ricci flow has been given so much attention in the past few years. Since then, it has been used to prove other major theorems, such as the Differentiable Sphere Theorem in 2008 [1].

3. The Heat Equation

We start our journey in more grounded territory. In order to understand the Ricci flow in the context of heat flows, we must first understand heat flows.

3.1. **Intuition.** If we have some heat distribution, such as a plate with hot mashed potatoes on one side and ice cream on the other (a very strange

[3]The "2 dimensional Poincare conjecture" was already known via the classification of Riemann surfaces

meal), the heat from the mashed potatoes will dissipate into the plate and the ice cream and over time the temperature distribution will converge to a mush of constant temperature. Heat tends to dissipate from an ordered distribution into a disordered mush[4]. If there is no heat source the hottest points will cool down while the coldest ones warm up. This phenomena has been studied using the heat equation, which was introduced by Joseph Fourier. This equation says that a heat distribution u will evolve over time via the equation[5] $\frac{\partial u}{\partial t} = \Delta u$

3.2. **The Laplacian.** The above equation involves the Laplacian Δu. Many of you will remember this from vector calculus as being $\nabla \cdot (\nabla u)$ (or div(grad u)) but it is illustrative to think of it in terms of coordinates, where we have (in $\mathbb{R}^3$) $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$.

However, for the context of the Ricci flow, there is a third perspective that is even better. We can consider the Hessian matrix

$$H = \begin{bmatrix} \frac{\partial^2}{\partial x^2} & \frac{\partial}{\partial x}\frac{\partial}{\partial y} & \frac{\partial}{\partial x}\frac{\partial}{\partial z} \\ \frac{\partial}{\partial y}\frac{\partial}{\partial x} & \frac{\partial^2}{\partial y^2} & \frac{\partial}{\partial y}\frac{\partial}{\partial z} \\ \frac{\partial}{\partial z}\frac{\partial}{\partial x} & \frac{\partial}{\partial z}\frac{\partial}{\partial y} & \frac{\partial^2}{\partial z^2} \end{bmatrix}$$

Then from this we can see that $\Delta u = \operatorname{tr} H(u)$. This shows how we can think of the Laplacian as the trace of some matrix of second derivatives, which is a perspective that will help us understand Ricci curvature.

3.3. **The Heat Equation.** The heat equation is a partial differential equation given by the expression

$$\frac{\partial u}{\partial t} = \Delta u$$

We think of t as being a time parameter and the right hand side as being derivatives in space. Given an initial distribution of heat (once again, disregarding issues about the boundary), a solution to this equation will give the temperature of points at a time t. We call such a solution a "heat flow." To check our intuition that the hottest points should cool down, suppose at some time t_0 the point (x_0, y_0, z_0) was the hottest point. Then since $u(x_0, y_0, z_0)$ is at a maximum, $H(u)$ is non-positive definite so $\operatorname{tr} H(u)$ is non-positive so the temperature cannot be increasing. In order to show that the temperature is actually decreasing, we need a stronger maximum principle, but we will not derive one here.

[4]One can formalize this idea in terms of entropy but we will not do that here

[5]We are ignoring the boundary conditions here. In the context of the Ricci flow, we will be dealing with compact manifolds without boundary so this is not an issue

4. Riemannian Curvature

The second step in understanding a "heat flow of curvature" is to exploring the study of curvature of surfaces and other spaces. Unlike the idea of the curvature of a curve from calculus, the intrinsic curvature of a space lacks a simple definition and is much harder to understand intuitively.

4.1. **Intuitive idea.**

If we can all agree on one thing, it is that Euclidean space is flat. With this idea, we can try to see how the geometry of our space[6] differs from Euclidean space locally. One thing that we can observe is that in the plane, there are no bumps or deformations. We need some way to mathematically measure this.

We can do this in the following way. On our space[7], we will pick a point (the north pole) and a direction from that point. Then we move along a triangle south to the equator, then west a quarter of the way around the ball and then back north to the pole. If we translate our vector that we picked at the north pole along this path, we end up with a different vector.[8] For those of you who have studied spherical trigonometry, this is familiar; the larger a triangle on a sphere, the more it deviates from a flat triangle (note that the sum of the angles in our triangle add up to $\frac{3\pi}{4}$ instead of $\frac{\pi}{2}$)[9]. Note that if we do it this same procedure in the plane with straight lines, we always get the same vector back. This is the key idea that allows us to define curvature. We notice that if we translate our vector along a polygon of whatever the equivalent of straight lines are, we might change what vector we end up with. This is cause by the curvature of the sphere and we would like to do this on other spaces as well.

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[6]The spaces that we are considering are smooth manifolds, which locally smoothly resemble Euclidean space. There are plenty of topological spaces that are not manifolds, but differential geometry typically studies manifolds. In this talk we will use the word "space" for "manifold" to avoid introducing unnecessary terminology.

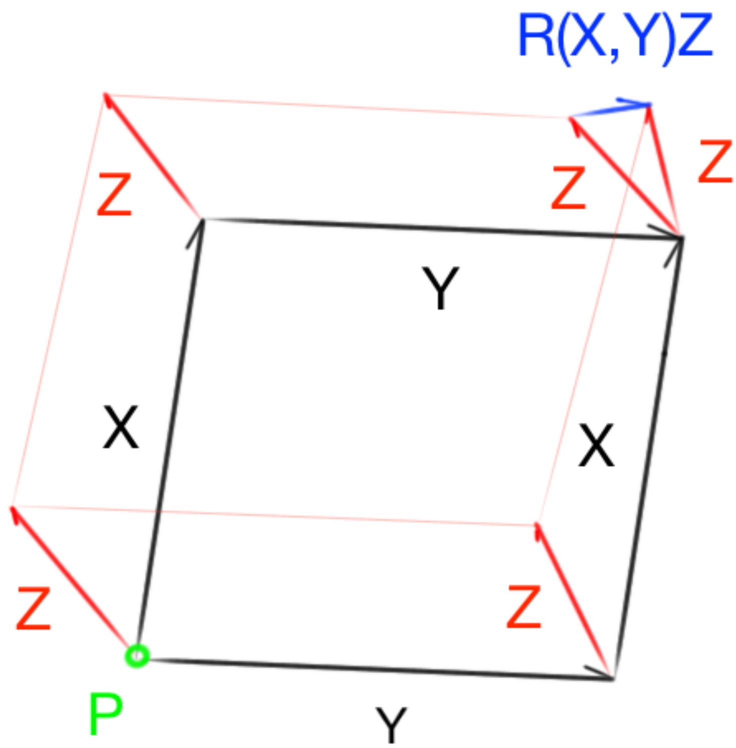
[7]I used an exercise ball in the talk, but you can use the picture of the earth instead

[8]Note that we are assuming the idea of parallel transport and that great circles are geodesics on the sphere.

[9]This leads us to the fact that two similar triangles on a sphere are necessarily congruent, which is not at all true in flat space

Image courtesy of Andrew Steele.[10]

4.2. **The Riemann Curvature Tensor.** Spheres are very nice spaces in that every point looks like every other point. However, earth is not exactly a sphere; the geometry of Mount Everest looks very little like that of the Marianas trench so we want some way to define curvature locally on spaces that are not symmetric. To make sense of curvature on such a space, instead of using a giant triangle like on our sphere, we use a small (infinitesimal) square. To do this, we need to pick three vectors at a point P in our space. These three directions will be the first direction that we translate along (which we call X), the second direction we translate along (Y), and the vector we translate(Z). With all of these vectors, we end up getting a vector which is the difference between the first vector and the final vector we get. Of course, if we transport along a tiny, tiny square, the difference in the vectors goes to zero so we need to normalize by the area of the square (or the side length squared in the limit). This is a diagram of what the curvature tensor $R(X, Y)Z$ is.



We use this idea to define the Riemann curvature tensor, which we are going to think of as a map that eats three vectors and produces a fourth vector[10]. What it does is translates a vector Z along an infinitesimal square defined by two vectors X and Y. In order to translate a vector Z, we take the derivative of the vector Z along X, then the derivative of that along Y. Then, in order to translate around a square, [11] we use the formula

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[X,Y]} Z$$

That is a lot to process. Let's break it down a bit.

We should think of the ∇_X and the ∇_Y as derivatives in the X and Y directions. We will ignore the last term, which vanishes when X and Y

[10]The word "tensor" just means that the map is linear in each of its arguments.

[11] We are ignoring the question as to why these translations form a closed square, which has to do with the precise definition of how we take derivatives of vectors. The precise reason is that the Levi-Civita connection is torsion-free.

are orthonormal vector fields[12]. In flat Euclidean space, the fact that the curvature tensor is zero is exactly the same as a fact that we are much more familiar with: the equality of mixed partials!

The Riemann curvature tensor completely determines how the shape of a manifold differs from flat space although it is a painful calculation to prove this [6]. The key point is that if the curvature tensor is identically zero then the space is locally isometric to flat space and if the tensor is non-zero than it is not. Note that on our cylinder, we also always get the same vector back so cylinders are flat (this shows that intrinsic curvature is a little more subtle than it might seem) However, paper is flat and we can roll it into a cylinder so perhaps the fact that a cylinder is flat is not entirely surprising.

Also, note that the final product is a vector and that in some sense the Riemann curvature tensor as a geometric second derivative[13]. We can think of it as an analog to the Hessian except that it has 3 vectors as inputs and a vector as an output. In the same way that if the Hessian of a function is identically zero then it is linear, if the Riemann tensor is identically zero than the space is flat.

5. Ricci Curvature

So if we think of the curvature $R(\cdot,\cdot)\cdot$ as being a map that eats three vectors and produces a fourth vector, we can ask how much the vector that we produce agrees with the second vector. Due to various symmetries, this is the same as asking how much it agrees with the first vector and the question of how much it agrees with the third vector is trivial (it will always be zero). One way of expressing this is that if we decompose our vector space into the orthonormal basis e_i, then given vectors X and Z we can construct the quantity $Ric(X, Z)$. This is known as the Ricci tensor[14]

$$Ric(X, Z) = \sum_{i=1}^{n} R(X, e_i)Z \cdot e_i$$

where the $\cdot$ is the usual dot product. Keep in mind, this takes the vector Z and translates it along a bunch of squares whose first side is always X.

[12] This terms forces the map to be linear and hence to be well defined regardless of how we extend X and Y in a neighborhood

[13] This analogy is precise in the sense that if we expand out the metric in geodesic normal coordinates around a point, we have $g_{ij} = \delta_{ij} - \frac{1}{3}R_{ikjl}x^k x^l + O(|x|^3)$. We have to flat the last term of the curvature tensor to obtain this result.

[14] Once again, ignoring why it is called a tensor.

Then it determines how much the final answer agrees with the second side of the square[15].

If we write the Laplacian as the trace of the Hessian, we get $\Delta = \sum_{i=1}^{n} \frac{\partial}{\partial e_i} \frac{\partial}{\partial e_i}$ and this looks very similar to $Ric(X, Z) = \sum_{i=1}^{n} R(X, e_i)Z \cdot e_i$ if we think of the Riemann curvature tensor as being a geometric Hessian. In this sense, the Ricci curvature is a geometric Laplacian[16]. In the same way that the Laplacian determines how much the gradient flow of a function stretches and compresses, the Ricci flow has the nice property that it fully determines how volumes on our space differ from volumes in flat space.

6. The Ricci Flow

The Ricci flow is a flow that changes the shape of our space proportional to the Ricci curvature. We have not defined how we determine what the "shape" of a space is, but it turns out that it is completely determined by the behavior of the dot product. Therefore, the formula of the Ricci flow is the following. Given two stationary vector fields on our space, X and Y (that do not evolve in time), at every point, we define the Ricci flow as

$$\frac{\partial}{\partial t} X \cdot Y = -2Ric(X, Y)$$

Notice that this is a system of equations because we get a different equation for each X and Y. The vector fields are staying the same but their dot product is changing, which we think of as a change of the metric and so a change of the underlying shape of the space. This seems to be massively overdetermined but a key point is that everything is nice and linear so we can write this in terms of the basis vectors e_i and e_j where we get the finite system of equations

$$\frac{\partial}{\partial t} e_i \cdot e_j = -2Ric(e_i, e_j)$$

Now there are exactly as many equations as there are unknowns.

Of course, we cannot insist that e_i forms an orthonormal basis after time zero if the space is not flat.[17]. The factor of -2 appears because analysts

[15]It is worthwhile to note that the Ricci tensor is symmetric (i.e that $Ric(X, Z) = Ric(Z, X)$). This is important because in order for the Ricci flow to be defined, the deformation must be symmetric in order for the metric tensor to remain so.

[16]This is precise since, in harmonic coordinates, $Ric_{ij} = -\frac{1}{2}\Delta g_{ij} +$ lower order terms...

[17]We are avoiding the need to define coordinate charts but normally the equation is written as $\frac{\partial}{\partial t} g_{ij} = -2Ric_{ij}$

and geometers cannot decide on the definition of the Laplacian and differ by a sign. However, the actual vector fields e_i are not evolving, just the dot product of e_i and e_j.

However, the key observation is that we should notice a parallel between this formula and the heat equation $\frac{\partial u}{\partial t} = \Delta u$. We have tried to justify that we can think of the Ricci curvature tensor as being analogous of Δ (well, -2Δ but close enough) and so the Ricci flow is a heat flow of shape. As a demonstration of this fact, `here is a demonstration` of the Ricci flow acting on a deformed two sphere. Notice how the shape becomes more and more spherical as time goes on. [5] In the same way that the heat equation spreads the heat evenly throughout the space, the Ricci flow spreads the curvature evenly throughout the space.

6.1. **Some caveats.** Relying back to our intuition about how heat behaves, we expect that the Ricci flow should smooth out our space and make it more uniform. In reality, the Ricci flow is more complicated than a heat flow. First off, it is nonlinear (the linear combination of two solutions is no longer a solution) because how we take derivatives on the space is determined by the shape of the space in the first place and there is a dot product in the definition of the Ricci tensor. Secondly, it behaves more like the reaction-diffusion equation $\frac{\partial}{\partial t} u = \Delta u + u^2$. The first term on the right hand side behaves as a diffusion term that disperses heat throughout the space whereas the second acts a reaction term that concentrates heat at a point. Reaction-diffusion equations can be thought as a tug-of-war between the diffusion process and the reaction process. If diffusion wins, the solution will smooth itself out much like the normal heat equation. If the reaction term wins out, the heat will become more and more intense and can sometimes even become become infinite in a finite amount of time![18] With the Ricci flow, the first key estimate is that the curvature grows similarly to $\frac{\partial}{\partial t} u = \Delta u + C u^2$ and that sometimes the reaction term wins out. In this case, the curvature becomes larger and larger until the shape tears itself apart (or shrinks to a point). This is part of what made analysis of the flow so difficult. It is not obvious when singularities occur or what they look like[19]. In any case, the

[18] Like the equation $\frac{\partial}{\partial t} u = u^2$.

[19] In three dimensions, one of Perelman's key contributions on his way to Poincaré was to classify all the possible structures of singularities and to show that a singularity known as the "cigar soliton" would not develop.

Ricci flow is a powerful tool that can be used to prove classification theorems that were unable to be solved using more conventional techniques.

7. Technical Notes

This section is written for those who are more familiar with geometric analysis. It addresses some of the points that we are skimming or ignoring. Some of the details have already been discussed in the footnotes.

7.1. **Introduction and History.**

The Ricci flow equation is not parabolic so it is not a heat flow[20] . As such, establishing the existence was a major accomplishment. Hamilton was able to establish existence via a technical argument [3] but a few years later Dennis DeTurck discovered the DeTurck trick, which is a much more accessible argument. This trick found a related flow that is parabolic and then used the diffeomorphism-invariance to establish existence for the standard Ricci flow [12]. Uniqueness then follows from a maximum principle argument.

Once existence and uniqueness were established, Hamilton studied the behavior of the flow. He found that the Ricci flow does not simply behave like a diffusion equation. Instead, it is more like a reaction-diffusion equation which can form singularities. In order to have any hope of getting the topological results that he proposed, one would have to understand the formation of singularities and their relationship to solitons of the flow. Although Hamilton was not able to gain a full understanding of this phenomena, he was able to use the flow to show that under positive curvature assumptions, the Poincaré conjecture holds. He did so by showing that in this case the flow would take a simply connected space to a round singularity, and hence the space is diffeomorphic to a sphere.

Over the next 20 years, various mathematicians contributed to the understanding of the Ricci flow. Versions of existence and uniqueness theorems were established for non-compact manifolds as well as theorems exploring the geometry of possible singularities. Perelman's first breakthrough is the discovery of relevant functionals which allow us to consider the Ricci flow as a gradient flow. He produced several entropy inspired functionals that are monotonic under the flow. These were used to establish and slowly improve

[20]The diffeomorphism invariance of the flow introduces negative terms into its symbol [12].

the estimates on the Ricci flow[21]. These were ultimately used to control the formation of singularities well enough to perform surgery on any developing singularity. He also proved that one only needed to perform surgery finitely many times to create pieces that approach a model geometry under a flow. The technical details are far outside the scope of this talk. This proof uses seemingly every known technique in nonlinear analysis and proves not only the Poincaré conjecture [4], but also Thurston's Geometrization Conjecture. This is geometric analysis' most important success and arguably the most important theorem in geometry from the past century.[22]

Note that we are using the word "space" to mean "compact smooth manifold." We are doing this in order to introduce unnecessary terminology but for the sake of precision it should be noted.

7.2. **The Heat Equation.** The heat equation is parabolic, meaning that its symbol is positive definite aside from a degenerate direction (the t direction). In general, such equations have real analytic solutions for all positive time, even for relatively irregular boundary data[23]. The domain of the solution to the heat equation for a space M is generally $M \times [0, \infty)$ and the boundary is $M \times 0 \cup \partial M \times [0, \infty)$. For this, we are really interested in compact manifolds without boundary, so to get a solution we must only specify the initial distribution on $M \times 0$. Such solutions will have a strong maximum principle and are very well behaved. In fact, one can show that this is a gradient flow of Dirichlet energy. For nice initial distributions (bounded energy) solutions will converge smoothly in time towards a constant solution. This is true on a compact Riemannian manifolds as well, although it is somewhat harder to define the Laplacian. This idea gets a lot of use in heat theoretic approaches to many theorems. [6]

7.3. **Riemannian Curvature.** The definition of the Riemannian curvature is $R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[X,Y]} Z$. We are using ∇_X to denote

[21] He used groundbreaking work from an astonishing number of fields, including Alexandrov geometry, surgery theory, functional analysis and several other fields to obtain these estimates.

[22] In his partial exposition of the proof, Terence Tao writes that even if the result were solely in the domain of partial differential equations and had no applications to geometry and topology, it would still be "the most technically impressive and significant result in the field of nonlinear PDE in recent years." [11]

[23] If the equation is highly non-linear more conditions may have to be imposed but this is a good heuristic

a covariant derivative in the X direction with respect to the Levi-Civita connection.

We are completely overlooking the fact that in general there is not a unique way of taking derivatives of a vector on a manifold. Once must introduce the idea of a connection and use one in order to define parallel transport. In this talk we are using the Levi-Civita connection, which is the unique torsion-free metric compatible connection on a Riemannian manifold. In the talk I did not define the metric tensor, and used the phrase "dot product" instead. I thought that this would convey the essential idea without being too technical.

7.4. **Ricci Curvature.** The formal definition of the Ricci curvature in coordinates is that given two vector fields X and Z, $Ric(X, Z) = \sum_{i=1}^{n} \langle R(X, e_i)Z, e_i \rangle$. It is the contraction of the Riemann curvature tensor along second and last indices. In three dimensions, the Ricci curvature has the nice property that it completely determines the entire curvature tensor, but that fails in higher dimensions. In harmonic coordinates, one can write that $Ric_{ij} = -\frac{1}{2}\Delta g_{ij} +$ lower order terms..., which shows the precise sense in which it is a Laplacian up to lower order terms.

The Ricci curvature determines how the volume of small cones around a geodesic differs from flat space. The tensoriality and symmetry of the Ricci curvature follows from the tensoriality and various symmetries of the full curvature tensor.

7.5. **Ricci Flow.** We are not introducing coordinate charts which is why we have given such an awkward definition of the Ricci flow. However, the definition of the Ricci flow is $\partial_t g_{ij} = -2Ric_{ij}$. From the previous section, we can write

$$Ric_{ij} = -\frac{1}{2}\Delta g_{ij} + \text{lower order terms...}$$

and so we find that the Ricci flow can be written (point-wise) as

$$\partial_t g_{ij} = \Delta g_{ij} + \text{lower order terms...}$$

In this sense, the Ricci flow can be viewed as a non-linear reaction diffusion equation.

The reaction term in the Ricci flow can be bounded quadratically [12]. The Ricci flow often does blow up much like the associated reaction diffusion equation. It is not hard to prove that once you have short term existence, you have existence until the Riemannian curvature blows up [8] so a singularity is generally defined to be a blow up of the Riemannian curvature tensor.

However, it has been shown that at any singularity the Ricci curvature blows up as well [7]. It is unknown whether the scalar curvature necessarily blows up at a singularity.

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