Solutions for the Gordon Prize Examination

1. Arrange the numbers in the array:

$$\begin{array}{cccccccc} 2001 & 2000 & \cdots & 1668 & 1667 & 1666 & \cdots & 1335 \\ 1001 & 1000 & \cdots & 668 & 1334 & 1333 & \cdots & 1002 \\ 1 & 3 & \cdots & 667 & 2 & 4 & \cdots & 666 \end{array}$$

The first row consists of the numbers 2001 to 1335 in decreasing order. In the second row, the first 334 numbers are 1001 to 668, and the last 333 numbers are 1334 to 1002. In the last row, the first 334 numbers are the odd numbers from 1 to 667, and the last 333 numbers are the even numbers from 2 to 666. Each column has sum 3003.

2. Let us "color" the squares of the chessboard with the "colors" a, b and c as in the picture. Then, independently of the path a dolphin follows starting from the lower lefthand corner square, the sequence of the colors of the squares along its path will be $a, b, c, a, b, c, \ldots$ Therefore the number of a's the dolphin meets is not less than the number of b's. But there are 21 a's and 22 b's on the chessboard, so the dolphin cannot visit all b's.

b	c	a	b	c	a	b	c
a	b	c	a	b	c	a	b
c	a	b	c	a	b	c	a
b	c	a	b	c	a	b	c
a	b	c	a	b	c	a	b
c	a	b	c	a	b	c	a
b	c	a	b	c	a	b	c
a	b	c	a	b	c	a	b

3. First, note that there is a point A on the plane such that the distances from A to the lattice points on the plane are all different. Indeed, $A = (\frac{1}{3}, \sqrt{2})$ is such a point: $\operatorname{dist}\big((n,m), (\frac{1}{3}, \sqrt{2})\big) = \operatorname{dist}\big((k,l), (\frac{1}{3}, \sqrt{2})\big)$ for integers n, m, k, l implies $\big(n - \frac{1}{3}\big)^2 + \big(m - \sqrt{2}\big)^2 = \big(k - \frac{1}{3}\big)^2 + \big(l - \sqrt{2}\big)^2$, so $r - 2m\sqrt{2} = s - 2l\sqrt{2}$ where r and s are rational, so $m = l$, so $|n - \frac{1}{3}| = |k - \frac{1}{3}|$, so $n = k$.
Now, let $\alpha_1 < \alpha_2 < \alpha_3 < \ldots$ be the ordered list of all distances from A to lattice points in the plane. Choose any real number R with $\alpha_{2001} < R < \alpha_{2002}$. Then the disc of radius R centered at A contains exactly 2001 lattice points.

4. Note that all roots of P are negative: since $a_{n-1}, \ldots, a_1 \geq 0$, $P(x) > 0$ for nonnegative x. Therefore, $P(x) = \prod_{i=1}^{n} (x + \alpha_i)$ with $\alpha_1, \ldots, \alpha_n > 0$. By the arithmetic-geometric mean inequality, for all $i = 1, \ldots, n$ we have $\frac{2+\alpha_i}{3} = \frac{1+1+\alpha_i}{3} \geq \sqrt[3]{\alpha_i}$ and so, $2 + \alpha_i \geq 3\sqrt[3]{\alpha_i}$. Also, $\prod_{i=1}^{n} \alpha_i = \prod_{i=1}^{n} (0 + \alpha_i) = P(0) = 1$. Hence,

$$P(2) = \prod_{i=1}^{n} (2 + \alpha_i) \geq \prod_{i=1}^{n} 3\sqrt[3]{\alpha_i} = 3^n \sqrt[3]{\prod_{i=1}^{n} \alpha_i} = 3^n.$$

For problems 5 and 6 see the solutions of the corresponding problems from the Rasor-Bareis exam.