Solutions: 2002 Gordon Prize Examination

1. The series

$$1-\frac{1}{2}-\frac{1}{2}+\frac{1}{\sqrt[3]{2}}-\frac{1}{2}\cdot\frac{1}{\sqrt[3]{2}}-\frac{1}{2}\cdot\frac{1}{\sqrt[3]{2}}+\frac{1}{\sqrt[3]{3}}-\frac{1}{2}\cdot\frac{1}{\sqrt[3]{3}}-\frac{1}{2}\cdot\frac{1}{\sqrt[3]{3}}\cdots$$

converges to 0, but the series

$$1^3-\left(\frac{1}{2}\right)^3-\left(\frac{1}{2}\right)^3+\left(\frac{1}{\sqrt[3]{2}}\right)^3-\left(\frac{1}{2}\cdot\frac{1}{\sqrt[3]{2}}\right)^3-\left(\frac{1}{2}\cdot\frac{1}{\sqrt[3]{2}}\right)^3+\left(\frac{1}{\sqrt[3]{3}}\right)^3-\left(\frac{1}{2}\cdot\frac{1}{\sqrt[3]{3}}\right)^3-\left(\frac{1}{2}\cdot\frac{1}{\sqrt[3]{3}}\right)^3\cdots$$
$$=1-\frac{1}{8}-\frac{1}{8}+\frac{1}{2}-\frac{1}{8}\cdot\frac{1}{2}-\frac{1}{8}\cdot\frac{1}{2}+\frac{1}{3}-\frac{1}{8}\cdot\frac{1}{3}-\frac{1}{8}\cdot\frac{1}{3}\cdots$$

diverges to $+\infty$.

2. Relatively prime integers have no prime factor in common, and any integer ≥ 2 has at least one prime factor. Therefore the fifteen integers in S have pairwise disjoint sets of prime factors. The fifteenth prime is 47, so there is an element s of S whose smallest prime factor is ≥ 47. Since $47^2 > 2002$, it can have no other prime factors, and therefore s is prime.

3. We have $A = (1/2)ab\sin\gamma$, so $ab = 2A/\sin\gamma$, and so

$$c^2 = a^2 + b^2 - 2ab\cos\gamma = (a+b)^2 - 2ab(1+\cos\gamma) = (a+b)^2 - 4A(1+\cos\gamma)/\sin\gamma.$$

If a product ab is fixed, then the sum $a+b$ is minimum when $a = b$. So c is minimum when $a = b = \sqrt{2A/\sin\gamma}$.

4. SOLUTION I
Since there are finitely many points, there are finitely many pairs of points, so the line segments joining pairs of the points have only finitely many directions. Choose a direction not among these. Set up Cartesian coordinates so that the y-axis has this direction. That means that the x-coordinates of the points are all different. Number the points according to increasing x-coordinates: $A_k = (x_k, y_k)$ and $x_1 < x_2 < \cdots < x_{2n}$. Let the first line segment join A_1 to A_2, the second line segment join A_3 to A_4, and so on. These line segments cannot cross each other, because points from different line segments have different x-coordinates.
SOLUTION II
Induction on n. If $n = 1$, just join the two points by a line segment. Assume the result for $n \geq 1$ and let $A_1, \cdots, A_{2(n+1)}$ be distinct points in the plane. The boundary of the convex hull of the points is a polygon. Let A_j be a vertex of that polygon and let A_i be one of the other points on the polygon nearest A_j. Join A_i and A_j by a line segment. By induction, the $2n$ remaining A_k can be joined in disjoint pairs by line segments. Each of these line segments is disjoint from that joining A_i and A_j. Hence there are $n+1$ pairwise disjoint line segments joining pairs of points of $A_1, \cdots, A_{2(n+1)}$.
SOLUTION III
If $I_1, I_2, \cdots, I_n$ are any n segments connecting the points $A_1, \cdots, A_{2n}$ in pairs, let $L(I_1, \cdots, I_n)$ be the sum of the lengths of $I_1, \cdots, I_n$. Let z be the minimum of all possible $L(I_1, \cdots, I_n)$ and let $J_1, \cdots, J_n$ be a set of segments which correspond to this minimum (it does not have to be unique). We claim that for any $i \neq k$, segments J_i and J_k do not cross. Indeed, if they *did* cross, we could make $L(J_1, \cdots, J_n)$ even smaller by replacing J_i and J_k by two new segments as in the picture.

Problems 5, 6: see the Rasor-Bareis solutions.