Gordon Solutions

1. Given any selection of 1004 distinct integers from the set $\{1, 2, \ldots, 2004\}$, show that some three of the selected integers have the property that one is the sum of the other two.

Solution I. Let m be the largest of the selected integers. This leaves 1003 selected integers in $\{1, \ldots, m-1\}$. Consider the pairs of distinct integers in $\{1, \ldots, m-1\}$ that add to m: these pairs are $(1, m-1)$, $(2, m-2)$, $(3, m-3)$, etc. If m is even, then there are $(m-2)/2$ pairs, and one number $m/2$ left over. If m is odd, there are $(m-1)/2$ pairs with nothing left over. There are 1003 selected integers in $\{1, \ldots, m-1\}$, and $1003 = (2006-2)/2+1 > (m-2)/2+1$ if m is even and $1003 = (2007-1)/2 > (m-1)/2$ if m is odd, so at least one of the pairs has both components selected. This pair, together with m, gives us a selected triple such that one of the integers is the sum of the other two.

Solution II. Let A be the set of selected integers and let m be the largest element in A. Let $B = \{m - a \mid a \in A, a \neq m\}$. Then $|A| = 1004$ and $|B| = 1003$. Hence $A \cap B$ contains at least 3 elements, say a, b, and c. So, for some x, y, and $z \in A$, we have

$$(*) \qquad a = m - x,\ b = m - y, \text{ and } c = m - z.$$

None of a, b, or c can equal m. Also, since the a, b, and c are distinct, only one of them can equal $m/2$. Hence at least two(!) of the equations in $(*)$ involve three distinct elements of A (these two equations can be the same equation written in different order).

2. What is the greatest integer less than or equal to $\dfrac{1}{e^{\frac{1}{2004}}-1}$?

We estimate $e^{\frac{1}{2004}}$ using the Taylor series for e^x. Write $x = \frac{1}{2004}$, then $x > 0$ so

$$e^x - 1 = x + \sum_{k=2}^{\infty} \frac{x^k}{k!} > x, \qquad \text{so} \qquad \frac{1}{e^{\frac{1}{2004}}-1} < \frac{1}{\frac{1}{2004}} = 2004.$$

On the other hand,

$$e^x - 1 = x + \sum_{k=2}^{\infty} \frac{x^k}{k!} < x + \sum_{k=2}^{\infty} x^k = x + \frac{x^2}{1-x} = \frac{x}{1-x}, \qquad \text{so}$$

$$\frac{1}{e^{\frac{1}{2004}}-1} > \frac{1-\frac{1}{2004}}{\frac{1}{2004}} = 2003.$$

Therefore, the greatest integer less than or equal to $\dfrac{1}{e^{\frac{1}{2004}}-1}$ is 2003.

3. Let A, B be $n \times n$ matrices with real coefficients such that A is invertible. Is it possible that $AB - BA = A$?

Solution I. It is not possible. Assume $AB - BA = A$. Now B has an eigenvalue λ, possibly complex, so there is a nonzero complex vector u with $Bu = \lambda u$. Then $BAu = (AB - A)u = A(\lambda u) - Au = (\lambda - 1)Au$. Now A is invertible, so Au is not zero. Therefore $\lambda - 1$ is also an eigenvalue of B. Repeating the argument shows $\lambda - 2$, $\lambda - 3$, etc. are all eigenvalues of B. But this is impossible, since an $n \times n$ complex matrix has at most n eigenvalues.

Solution II. It is not possible. Assume $AB - BA = A$. Multiply both sides on the right by A^{-1} to get: $ABA^{-1} - B = I$. Matrices ABA^{-1} and B are similar, and therefore have the same trace. The trace of $ABA^{-1} - B$ is 0, but the trace of I is n. This contradiction shows the assumption $AB - BA = A$ was wrong.

4. Show that, for $a > 0$, $\displaystyle\int_{\frac{1}{2004}}^{2004} \frac{x-1}{1+ax+ax^2+x^3}\,dx = 0.$

Solution I. If $f(x) = (x-1)/(1+ax+ax^2+x^3)$, note that $f(1/x) = -x^2 f(x)$. Now for $u > 0$ write $\phi(u) = \int_{1/u}^{u} f(x)\,dx$. Then we may compute the derivative (using the Fundamental Theorem of Calculus and the Chain Rule):

$$\phi'(u) = f(u) - f\left(\frac{1}{u}\right)\cdot\left(\frac{-1}{u^2}\right) = f(u) - f(u) = 0.$$

Therefore ϕ is constant. But $\phi(1) = \int_1^1 f(x)\,dx = 0$, so we have $\phi(u) = 0$ for all $u > 0$, and in particular $\phi(2004) = 0$.

Solution II. Using the substitution $y = 1/x$ we see that

$$\int_{1/2004}^{1} f(x)dx = -\int_{1}^{2004} f(y)dy \qquad \text{and so}$$

$$\int_{1/2004}^{2004} f(x)dx = -\int_{1}^{2004} f(y)dy + \int_{1}^{2004} f(x)dx = 0.$$

Solution III. The partial fraction expansion is

$$\frac{x-1}{(x+1)(x^2+(a-1)x+1)} = \frac{\frac{2}{a-3}}{x+1} + \frac{\left(\frac{-2}{a-3}\right)x + \left(\frac{1-a}{a-3}\right)}{x^2+(a-1)x+1}$$

and the numerator of the second one is a constant times the derivative of the denominator, so

$$\int \frac{x-1}{x^3+ax^2+ax+1}\,dx = \frac{2}{a-3}\ln(x+1) + \frac{-1}{a-3}\ln\left(x^2+(1-a)x+1\right).$$

Substitute $x = 2004$ and $x = 1/2004$ for the integral:

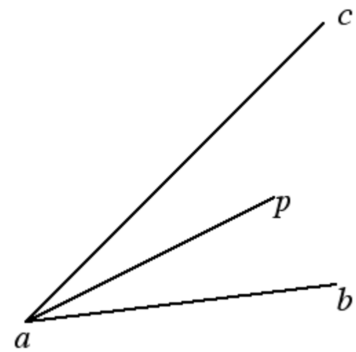
$$\int_{1/2004}^{2004} \frac{x-1}{x^3+ax^2+ax+1}\,dx$$

$$= \frac{1}{a-3}\Bigg(2\ln(2005) - 2\ln\left(\frac{2005}{2004}\right) - \ln\left(2004^2+(a-1)2004+1\right)$$

$$+\ln\left(\frac{1+(a-1)2004+2004^2}{2004^2}\right)\Bigg) = 0.$$

5. Let a, b and c be complex numbers forming a triangle in the complex plane. Show that there is a complex number p such that all of the following numbers are real:

$$\frac{(a-b)(a-c)}{(a-p)^2}, \qquad \frac{(b-a)(b-c)}{(b-p)^2}, \qquad \frac{(c-a)(c-b)}{(c-p)^2}.$$



The **argument** of a nonzero complex number z is the angle $\arg(z)$ that the vector from 0 to z makes with the real axis. When complex numbers are multiplied, their arguments add; when complex numbers are divided, their arguments subtract; if a complex number has argument 0, then it is real.

Now imagine an angle b, a, c in the complex plane. If segment a, p bisects the angle, then $\arg((a-b)(a-c)/(a-p)^2) = \arg(b-a) + \arg(c-a) - 2\arg(p-a) = 0 \mod 2\pi$ so $(a-b)(a-c)/(a-p)^2$ is real. As is well-known, the three angle-bisectors of a triangle intersect in a point, so that point is our solution.

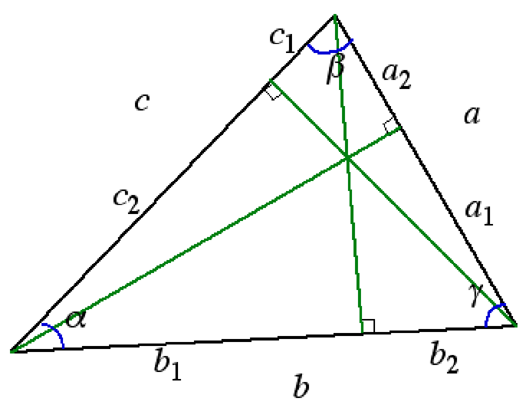
If the points a, b, c are collinear, they form a *degenerate* triangle, but then any point p in the line they span (other than a, b, c) will solve the problem.

6. Let α, β, γ be the angles of a triangle. Show that $\cos\alpha \cdot \cos\beta \cdot \cos\gamma \le \frac{1}{8}$.

Solution I. (*Submitted by Donald Seelig*) First, if the triangle is obtuse then $\cos\alpha \cdot \cos\beta \cdot \cos\gamma < 0$ and if the triangle is right, then $\cos\alpha \cdot \cos\beta \cdot \cos\gamma = 0$. So assume the triangle is acute.

The altitude from the vertex with angle α divides the opposite side a into two parts a_1, a_2. The altitude from the vertex with angle β divides the opposite side b into two parts b_1, b_2. The altitude from the vertex with angle γ divides the opposite side c into two parts c_1, c_2. From right triangle trigonometry, we get

$$\cos\alpha = \frac{b_1}{c} = \frac{c_2}{b}, \qquad \cos\beta = \frac{a_2}{c} = \frac{c_1}{a}, \qquad \cos\gamma = \frac{b_2}{a} = \frac{a_1}{b}.$$



Then $a_1 b_1 c_1 = abc\cos\alpha\cos\beta\cos\gamma = a_2 b_2 c_2$, and algebraic manipulation gives us:

$$\frac{1}{\cos\alpha\cos\beta\cos\gamma} = 2 + \frac{a_2}{a_1} + \frac{a_1}{a_2} + \frac{b_2}{b_1} + \frac{b_1}{b_2} + \frac{c_2}{c_1} + \frac{c_1}{c_2}.$$

Now note that for $x > 0$ we have $x + 1/x \ge 2$ (this follows from $(x-1)^2 \ge 0$), so

$$\frac{1}{\cos\alpha\cos\beta\cos\gamma} \ge 2 + 2 + 2 + 2 = 8.$$

Solution II. Note that

$$\cos\gamma = \cos(\pi - (\alpha+\beta)) = -\cos(\alpha+\beta),$$
$$2\cos\alpha\cos\beta = \cos(\alpha-\beta) + \cos(\alpha+\beta).$$

Hence:

$$\begin{aligned} &8\cos\alpha\cos\beta\cos\gamma - 1 \\ &\quad = -4\cos(\alpha+\beta)\big[\cos(\alpha+\beta) + \cos(\alpha-\beta)\big] - \big[\cos^2(\alpha-\beta) + \sin^2(\alpha-\beta)\big] \\ &\quad = -\big[2\cos(\alpha+\beta) + \cos(\alpha-\beta)\big]^2 - \sin^2(\alpha-\beta) \le 0. \end{aligned}$$

Solution III. For any real θ, the maximum of the function

$$f(x) = \cos(x)\cos(\theta - x) = \frac{1}{2}\big[\cos(\theta) + \cos(2x - \theta)\big]$$

is reached when $2x - \theta = 0$, i.e. when $x = \theta - x$. Thus, when α, β, γ are angles of a triangle, for any fixed γ the maximum of $\cos(\alpha)\cos(\beta)\cos(\gamma) = \cos(\alpha)\cos(\pi - \gamma - \alpha)\cos(\gamma)$ is reached when $\alpha = \pi - \gamma - \alpha = \beta$. Hence, the maximum of $F(\alpha,\beta,\gamma) = \cos(\alpha)\cos(\beta)\cos(\gamma)$ is reached when $\alpha = \beta = \gamma = \pi/3$. (If, say, $\beta \ne \gamma$, then $F(\alpha, (\beta+\gamma)/2, (\beta+\gamma)/2) > F(\alpha,\beta,\gamma)$.)