1. A spherical cherry of radius $R$ is dropped into a glass of the form $z = (x^2 + y^2)^2$. Find the maximum $R$ for which the cherry will reach the bottom of the glass.

Solution. Observe that because of the rotational symmetry the problem is equivalent to the following planar one: What is the maximal radius $R$ of a circle that can be placed above the graph of $z = x^4$, touching it at the point $(0,0)$. Clearly the center of the circle can be assumed to be on the $z$ axis.

Thus we need to find the supremum of the set of those $R$ for which the system of equations

$$z = x^4, \quad x^2 + z^2 - 2Rz = 0$$

has no solutions other than $(0,0)$.

Substituting in the second equation $z$ with $x^4$, we get $x^2 + x^8 - 2Rx^4 = 0$, or $1 + x^6 - 2Rx^2 = 0$; thus we need to find the supremum of the set of $R$ for which the equation

$$1 + x^6 - 2Rx^2 = 0 \quad (\ast)$$

has no solutions. From this equation, $R = \frac{1}{2}(x^{-2} + x^4)$; the function $\frac{1}{2}(x^{-2} + x^4)$ attains its minimum at the point $x = \frac{1}{\sqrt{2}}$, and this minimum is equal to $\frac{1}{2}\left(\sqrt{2} + \frac{1}{\sqrt{2}}\right) = \frac{3}{4}\sqrt{2}$. Hence, the equation $(\ast)$ has solutions for $R > \frac{3}{4}\sqrt{2}$ and has no solution for $R < \frac{3}{4}\sqrt{2}$; so, the maximum $R$ for which “the cherry will reach the bottom of the glass” is $\frac{3}{4}\sqrt{2}$.

Another approach: $R$ is determined by the fact that the polynomial $t^3 - 2Rt + 1$ has a multiple positive root. Thus, as a multiple root of a polynomial is also a root of its derivative, we need to solve the system $t^3 - 2Rt + 1 = 0, 3t^2 - 2R = 0$. We get $t^3 = \frac{1}{2}$, and $R = \frac{3}{4}\sqrt{2}$.

2. Is there a differentiable function $f$ on $(0, \infty)$ satisfying $f'(x) = f(x + 1)$ for all $x$ and such that $\lim_{x \to \infty} f(x) = \infty$?

Solution. No, such a function cannot exist. Assume that $f'(x) = f(x + 1)$ for all $x$ and $\lim_{x \to \infty} f(x) = +\infty$ (if $\lim_{x \to \infty} f(x) = -\infty$, replace $f$ by $-f$). Then for $x$ large enough $f$ is positive, so $f'$ is positive, so $f$ is increasing, so $f'$ is increasing. But for any $x$, by the mean value theorem, $f(x + 1) = f(x) + f'(c)$ for some $c \in [x, x + 1]$, so, $f'(x) = f(x) + f'(c) > f'(c)$ if $f(x) > 0$, so $f'$ cannot be increasing.
3. Let $a$ and $b$ be real numbers. Consider the power series (in powers of $x$) for the function $f(x) = e^{ax} \cos(bx)$. Show that the series either has no zero coefficients or has infinitely many zero coefficients.

Solution. Since $\cos bx = \Re e^{ibx}$, we have $f(x) = \Re e^{(a+bi)x} = \Re e^{cx}$, where $c = a+bi \in \mathbb{C}$. Now, since $e^{cx} = \sum_{n=0}^{\infty} \frac{1}{n!} c^n x^n$, the coefficients of the power series of $f$ are of the form $\frac{1}{n!} \Re (c^n)$, and it remains to show that $\Re (c^n) = 0$ either for no $n$ or for infinitely many $n$.

Let $\text{Arg}(c) = \theta$. For any $n \in \mathbb{N}$ we have $\text{Arg}(c^n) = n\theta$, so $\Re (c^n) = 0$ iff $n\theta = \frac{k}{2} \mod \pi$, that is, iff $n\alpha = \frac{1}{2} \mod 1$, where $\alpha = \frac{\theta}{\pi}$. If $\alpha$ is an irrational number, then $n\alpha \neq \frac{1}{2} \mod 1$ for all $n$. If $\alpha = \frac{k}{m}$ with $k, m \in \mathbb{Z}$ and $m$ is odd, then also $n\alpha = \frac{nk}{m} \neq \frac{1}{2} \mod 1$ for all $n$. Finally, if $\alpha = \frac{k}{m}$ where $k$ is odd and $m$ is even, then $n\alpha = \frac{nk}{m} = \frac{1}{2} \mod 1$ for all $n$ of the form $lm + \frac{1}{2}m$, $l \in \mathbb{Z}$, and so, for infinitely many $n$.

4. Show that there is no $2009 \times 2009$ matrix $A$ with rational entries such that $A^2 = 2I$, where $I$ is the identity matrix.

Solution 1. If $A$ is a $2009 \times 2009$ matrix such that $A^2 = 2I$, then $\det A^2 = 2^{2009}$, so $\det A = \pm 2^{1004} \sqrt{2}$ and is irrational. Hence, $A$ cannot have all rational entries.

Solution 2. $A$ has 2009 eigenvalues equal to $\sqrt{2}$ or $-\sqrt{2}$. Hence its trace, which is the sum of the eigenvalues, cannot be rational, contradicting that $A$ has rational entries.

5. Let $X$ be the square $[0,1] \times [0,1]$ in the plane, and let $|p-q|$ denote the distance between points $p, q \in X$. Suppose that $f : X \to X$ is a surjective contraction; prove that $f$ is actually an isometry.

Solution. Let the vertices of $X$ be $A_1A_2A_3A_4$. Consider the pair $A_1, A_3$ of opposite vertices, then $|A_1 - A_3| = \sqrt{2}$. Let $a_1, a_3 \in X$ be such that $f(a_1) = A_1$ and $f(a_3) = A_3$; then $|a_1-a_3| \geq |f(a_1) - f(a_3)| = \sqrt{2}$. Hence, $a_1$ and $a_3$ is also a pair of opposite vertices of $X$. The pair $A_2, A_4$ of opposite vertices is also an image under $f$ of a pair of opposite vertices, $a_2$ and $a_4$ respectively, which are different from $a_1$ and $a_3$.

There is an isometry (a rotation and/or reflection) $\phi$ of $X$ such that $\phi(a_i) = A_i$, $i = 1, 2, 3, 4$; consider the mapping $g = f \circ \phi^{-1}$. Then $g$ is also a surjective contraction, and $g$ preserves the vertices of $X$: $g(A_i) = f(\phi^{-1}(A_i)) = f(a_i) = A_i$, $i = 1, 2, 3, 4$. We will show that $g$ is the identity mapping, $g(q) = q$ for all $q \in X$; this will prove that $f = \phi$ and is an isometry.
We first claim that $g$ is the identity on the boundary of $X$. Take a point $p$ on the boundary of $X$; say, let $p \in [A_1, A_2]$. Let $|p - A_1| = d$; then $p$ is the unique point of $X$ with the property that $|p - A_1| \leq d$, $|p - A_2| \leq 1 - d$. Since $f(A_1) = A_1$ and $f(A_2) = A_2$, we have $|f(p) - A_1| \leq |p - A_1| \leq d$ and $|f(p) - A_2| \leq |p - A_2| \leq 1 - d$; hence, $f(p) = p$.

We now claim that $f$ is the identity on the interior of $X$. Take a point $q$ in the interior of $X$, let $p_1$ and $p_2$ be the orthogonal projections of $q$ to the opposite sides $[A_1, A_2]$ and $[A_3, A_4]$ of $X$, and let $|q - p_1| = d$. Then $q$ is the only point of $X$ with the property that $|q - p_1| \leq d$ and $|q - p_2| \leq 1 - d$. Since $f(p_1) = p_1$ and $f(p_2) = p_2$, we have $|f(q) - p_1| \leq |q - p_1| \leq d$ and $|f(q) - p_2| \leq |q - p_2| \leq 1 - d$; hence, $f(q) = q$.

6. Assume that your calculator is broken so that you can only add and subtract real numbers and compute their reciprocals. How can you use it to compute products?

Solution. First, let us observe that, given a real number $x$, our broken calculator allows us to compute $x^2$. Indeed, for $x \notin \{0, -1\}$, since $\frac{1}{x} - \frac{1}{x+1} = \frac{1}{x^2+x}$ we have

$$x^2 = (x^{-1} - (x+1)^{-1})^{-1} - x.$$

Note also that, given $x$, it is easy to calculate $x/2$:

$$\frac{x}{2} = (2x^{-1})^{-1} = (x^{-1} + x^{-1})^{-1}.$$

Now, to calculate $xy$, we can use the identity

$$xy = \left(\frac{x + y}{2}\right)^2 - \left(\frac{x - y}{2}\right)^2$$