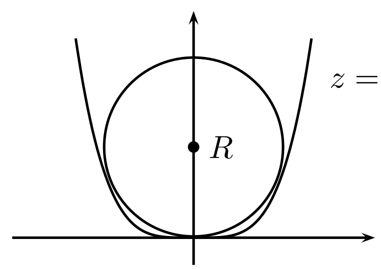
Solutions to 2009 Gordon Exam

1. A spherical cherry of radius R is dropped into a glass of the form $z = (x^2 + y^2)^2$. Find the maximum R for which the cherry will reach the bottom of the glass.

Solution. Observe that because of the rotational symmetry the problem is equivalent to the following planar one: *What is the maximal radius R of a circle that can be placed above the graph of $z = x^4$, touching it at the point* $(0, 0)$. Clearly the center of the circle can be assumed to be on the z axis.



Thus we need to find the supremum of the set of those R for which the system of equations

$$z = x^4, \qquad x^2 + z^2 - 2Rz = 0$$

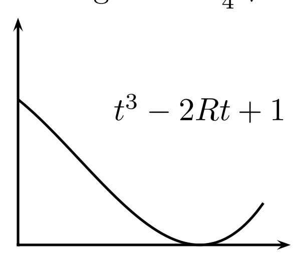
has no solutions other than $(0, 0)$.

Substituting in the second equation z with x^4, we get $x^2 + x^8 - 2Rx^4 = 0$, or $1 + x^6 - 2Rx^2 = 0$; thus we need to find the supremum of the set of R for which the equation

$$1 + x^6 - 2Rx^2 = 0 \tag{$*$}$$

has no solutions. From this equation, $R = \frac{1}{2}(x^{-2} + x^4)$; the function $\frac{1}{2}(x^{-2} + x^4)$ attains its minumum at the point $x = \frac{1}{\sqrt[6]{2}}$, and this minumum is equal to $\frac{1}{2}\big(\sqrt[3]{2} + \frac{1}{\sqrt[3]{2}}\big) = \frac{3}{4}\sqrt[3]{2}$. Hence, the equation $(*)$ has solutions for $R > \frac{3}{4}\sqrt[3]{2}$ and has no solution for $R < \frac{3}{4}\sqrt[3]{2}$; so, the maximum R for which "the cherry will reach the bottom of the glass" is $\frac{3}{4}\sqrt[3]{2}$.

Another approach: R is determined by the fact that the polynomial $t^3 - 2Rt + 1$ has a multiple positive root. Thus, as a multiple root of a polynomial is also a root of its derivative, we need to solve the system $t^3 - 2Rt + 1 = 0$, $3t^2 - 2R = 0$. We get $t^3 = \frac{1}{2}$, and $R = \frac{3}{4}\sqrt[3]{2}$.



2. *Is there a differentiable function f on $(0, \infty)$ satisfying $f'(x) = f(x + 1)$ for all x and such that* $\lim_{x\to\infty} f(x) = \infty$*?*

Solution. No, such a function cannot exist. Assume that $f'(x) = f(x + 1)$ for all x and $\lim_{x\to\infty} f(x) = +\infty$ (if $\lim_{x\to\infty} f(x) = -\infty$, replace f by $-f$). Then for x large enough f is positive, so f' is positive, so f is increasing, so f' is increasing. But for any x, by the mean value theorem, $f(x + 1) = f(x) + f'(c)$ for some $c \in [x, x + 1]$, so, $f'(x) = f(x) + f'(c) > f'(c)$ if $f(x) > 0$, so f' cannot be increasing.

3. *Let a and b be real numbers. Consider the power series (in powers of x) for the function $f(x) = e^{ax}\cos(bx)$. Show that the series either has no zero coefficients or has infinitely many zero coefficients.*

Solution. Since $\cos bx = \operatorname{Re} e^{ibx}$, we have $f(x) = \operatorname{Re} e^{(a+bi)x} = \operatorname{Re} e^{cx}$, where $c = a+bi \in \mathbb{C}$. Now, since $e^{cx} = \sum_{n=0}^{\infty} \frac{1}{n!} c^n x^n$, the coefficients of the power series of f are of the form $\frac{1}{n!}\operatorname{Re}(c^n)$, and it remains to show that $\operatorname{Re}(c^n) = 0$ either for no n or for infinitely many n.

Let $\operatorname{Arg}(c) = \theta$. For any $n \in \mathbb{N}$ we have $\operatorname{Arg}(c^n) = n\theta$, so $\operatorname{Re}(c^n) = 0$ iff $n\theta = \frac{\pi}{2} \bmod \pi$, that is, iff $n\alpha = \frac{1}{2} \bmod 1$, where $\alpha = \frac{\theta}{\pi}$. If α is an irrational number, then $n\alpha \neq \frac{1}{2} \bmod 1$ for all n. If $\alpha = \frac{k}{m}$ with $k, m \in \mathbb{Z}$ and m is odd, then also $n\alpha = \frac{nk}{m} \neq \frac{1}{2} \bmod 1$ for all n. Finally, if $\alpha = \frac{k}{m}$ where k is odd and m is even, then $n\alpha = \frac{nk}{m} = \frac{1}{2} \bmod 1$ for all n of the form $lm + \frac{1}{2}m$, $l \in \mathbb{Z}$, and so, for infinitely many n.

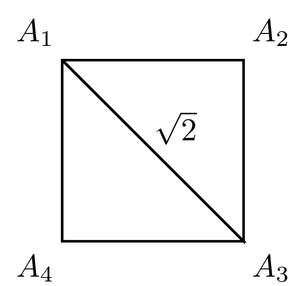
4. *Show that there is no 2009×2009 matrix A with rational entries such that $A^2 = 2I$, where I is the identity matrix.*

Solution 1. If A is a 2009×2009 matrix such that $A^2 = 2I$, then $\det A^2 = 2^{2009}$, so $\det A = \pm 2^{1004}\sqrt{2}$ and is irrational. Hence, A cannot have all rational entries.

Solution 2. A has 2009 eigenvalues equal to $\sqrt{2}$ or $-\sqrt{2}$. Hence its trace, which is the sum of the eigenvalues, cannot be rational, contradicting that A has rational entries.

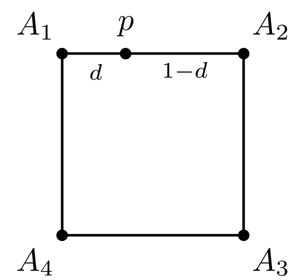
5. *Let X be the square $[0,1] \times [0,1]$ in the plane, and let $|p - q|$ denote the distance between points $p, q \in X$. Suppose that $f\colon X \longrightarrow X$ is a surjective contraction; prove that f is actually an isometry.*

Solution. Let the vertices of X be $A_1A_2A_3A_4$. Consider the pair A_1, A_3 of opposite vertices, then $|A_1 - A_3| = \sqrt{2}$. Let $a_1, a_3 \in X$ be such that $f(a_1) = A_1$ and $f(a_3) = A_3$; then $|a_1 - a_3| \geq |f(a_1) - f(a_3)| = \sqrt{2}$. Hence, a_1 and a_3 is also a pair of opposite vertices of X. The pair A_2, A_4 of opposite vertices is also an image under f of a pair of opposite vertices, a_2 and a_4 respectively, which are different from a_1 and a_3.

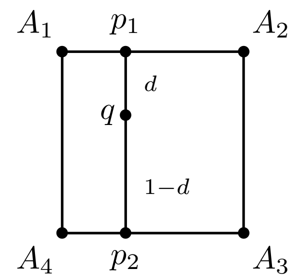


There is an isometry (a rotation and/or reflection) ϕ of X such that $\phi(a_i) = A_i$, $i = 1, 2, 3, 4$; consider the mapping $g = f \circ \phi^{-1}$. Then g is also a surjective contraction, and g preserves the vertices of X: $g(A_i) = f(\phi^{-1}(A_i)) = f(a_i) = A_i$, $i = 1, 2, 3, 4$. We will show that g is the identity mapping, $g(q) = q$ for all $q \in X$; this will prove that $f = \phi$ and is an isometry.

We first claim that g is the identity on the boundary of X. Take a point p on the boundary of X; say, let $p \in [A_1, A_2]$. Let $|p - A_1| = d$; then p is the unique point of X with the property that $|p - A_1| \leq d$, $|p - A_2| \leq 1 - d$. Since $f(A_1) = A_1$ and $f(A_2) = A_2$, we have $|f(p) - A_1| \leq |p - A_1| \leq d$ and $|f(p) - A_2| \leq |p - A_2| \leq 1 - d$; hence, $f(p) = p$.



We now claim that f is the identity on the interior of X. Take a point q in the interior of X, let p_1 and p_2 be the orthogonal projections of q to the opposite sides $[A_1, A_2]$ and $[A_3, A_4]$ of X, and let $|q - p_1| = d$. Then q is the only point of X with the property that $|q - p_1| \leq d$ and $|q - p_2| \leq 1 - d$. Since $f(p_1) = p_1$ and $f(p_2) = p_2$, we have $|f(q) - p_1| \leq |q - p_1| \leq d$ and $|f(q) - p_2| \leq |q - p_2| \leq 1 - d$; hence, $f(q) = q$.



6. *Assume that your calculator is broken so that you can only add and subtract real numbers and compute their reciprocals. How can you use it to compute products?*

Solution. First, let us observe that, given a real number x, our broken calculator allows us to compute x^2. Indeed, for $x \notin \{0, -1\}$, since $\frac{1}{x} - \frac{1}{x+1} = \frac{1}{x^2+x}$ we have

$$x^2 = (x^{-1} - (x+1)^{-1})^{-1} - x.$$

Note also that, given x, it is easy to calculate $x/2$:

$$\frac{x}{2} = (2x^{-1})^{-1} = (x^{-1} + x^{-1})^{-1}.$$

Now, to calculate xy, we can use the identity

$$xy = \left(\frac{x+y}{2}\right)^2 - \left(\frac{x-y}{2}\right)^2$$