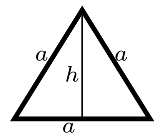
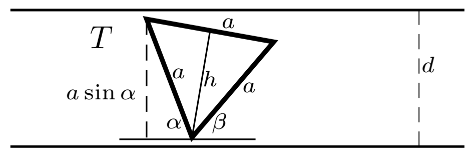
Solutions to 2010 Gordon Prize examination problems

1. *In the plane, consider an infinite strip of width d. Suppose every triangle of area 1 will fit inside the strip, after suitable translation and rotation. What is the minimum possible width d?*

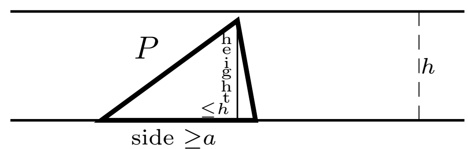
Solution. We claim that the minimum possible width d equals $\sqrt[4]{3}$, which is the height h of the equilateral triangle with area 1 (the triangle whose all sides are equal to $a = \frac{2}{\sqrt[4]{3}}$).



Indeed, if T is such a triangle that lies inside a strip of width d (see the picture), then since $\alpha + \beta + \pi/3 = \pi$, either $\alpha \geq \pi/3$ or $\beta \geq \pi/3$; if, say, $\alpha \geq \pi/3$, then $d \geq a \sin\alpha \geq a\sin(\pi/3) = h$.



On the other hand, for any triangle P of area 1, one of the sides of P has length $\geq a$. (If all sides of P have length $< a$, let γ be the minimal angle of P, so that $\gamma \leq \pi/3$; then area$(P) < \frac{1}{2}a^2 \sin\gamma \leq \frac{1}{2}a^2\sin(\pi/3) = 1$.) The corresponding height of P is $\leq \frac{2\,\text{area}(P)}{a} = \frac{2}{2/\sqrt[4]{3}} = \sqrt[4]{3} = h$, so P can be placed inside the strip of width h as in the following picture:



2. *Let ABC be a triangle with acute angles α, β and γ such that*

$$\tan(\alpha-\beta) + \tan(\beta-\gamma) + \tan(\gamma-\alpha) = 0.$$

Prove that ABC is isosceles.

<u>*Solution*</u>. Let $a = \tan\alpha$, $b = \tan\beta$, and $c = \tan\gamma$. Using the formula $\tan(x-y) = \frac{\tan x - \tan y}{1+\tan x \tan y}$, we get

$$\frac{a-b}{1+ab} + \frac{b-c}{1+bc} + \frac{c-a}{1+ca} = 0.$$

Hence,

$$(a-b)(1+bc+ac+abc^2) + (b-c)(a+ab+ac+a^2bc) + (c-a)(1+ab+bc+ab^2c) = 0$$

After opening brackets and canceling similar terms, we get $a^2c - a^2b + b^2a - b^2c + c^2b - c^2a = 0$. Now,

$$a^2c - a^2b + b^2a - b^2c + c^2b - c^2a = -a^2(b-c) + a(b^2-c^2) - bc(b-c) = (b-c)(-a^2+ab+ac-bc)$$
$$= (b-c)(a-b)(c-a)$$

So, either $a = b$, or $b = c$, or $c = a$, which implies that either $\alpha = \beta$, or $\beta = \gamma$, or $\gamma = \alpha$.

<u>*Another solution*</u>. Let $x = \alpha - \beta$, $y = \beta - \gamma$, and $z = \gamma - \alpha$, then $x + y + z = 0$ and $\tan x + \tan y + \tan z = 0$. Since $z = -(x+y)$ and $|z| < \pi/2$, we have

$$\tan(z) = -\tan(x+y) = \frac{-\tan(x) - \tan(y)}{1 - \tan(x)\tan(y)}.$$

So,

$$\tan(x) + \tan(y) + \tan(z) = \tan(x)\tan(y)\tan(z),$$

and we obtain that $\tan(x)\tan(y)\tan(z) = 0$. Hence, one of the angles x, y, or z is 0; without loss of generality, $x = 0$, so $\alpha = \beta$, and ABC is isosceles.

Yet another solution. Assume that ABC is not isosceles. Let $\alpha > \beta > \gamma$; put $x = \alpha - \beta$, $y = \beta - \gamma$, and $z = \alpha - \gamma$. Then $0 < x, y, z < \pi/2$, $z = x + y$, and we are also given that $\tan z = \tan x + \tan y$.

But tan is a strictly convex function on $[0, \pi/2)$, thus given two points a, b with $0 < a \le b < \pi/2$, the slope of the vector $(a, \tan a)$ is $\le \tan' a \le \tan' b$; thus the point $(b, \tan b) + (a, \tan a) = (a + b, \tan a + \tan b)$ lies strictly below the graph of the tangent, and it cannot be that $\tan(a + b) = \tan a + \tan b$.

3. *The number 2010 is written as a sum of two or more positive integers. What is the maximum possible product of these integers?*

Solution. There are only finitely many ways to decompose 2010 into a sum of positive integers, so there is a maximum value for the product of such a decomposition. Let $a_1, \dots, a_k$ be positive integers such that $a_1 + \dots + a_k = 2010$ and the product $P = \prod_{i=1}^{2010} a_i$ is maximal. Then

(i) none of a_i is 1, since if $a_i = 1$ for some i then we can replace the pair a_1, a_i by the singleton $a_1 + 1$, and thereby increase the product P;

(ii) none of a_i is greater or equal than 5, since if $a_i = 5$, we can replace a_i by the pair $a_i - 2, 2$ and increase P;

(iii) moreover, we can assume no a_i is equal to 4, since 4 can be replaced by the pair $2, 2$ without changing P;

(iv) at most two of a_i are equal to 2, since otherwise we can change $2, 2, 2$ to $3, 3$ and increase P.

So, the only possible combinations for which P is maximal are 3,3,3,...,3, or 2,3,3,...,3, or 2,2,3,...,3. But since 2010 is divisible by 3, the last 2 solutions do not come up, and the maximum possible product is 3^{670}.

4. *Let A be a* 2010×2010 *matrix such that in every row and in every column, exactly two entries are equal to 1 and the rest are 0. Prove that the determinant of A is either* 0 *or* $\pm 2^m$ *where m is even.*

Solution. The determinant of a matrix does not change, up to the sign, under permutation of rows or columns of a matrix, thus we are free to permute rows and columns of A. Permuting columns of A, we can move the 1s in the first row to the left side, so that the first line of A will become $(1\,1\,0\dots0)$. Then we find the row that contains 1 at the second column, permute it with the second line, and, if the another 1 in this row is not at the first column, move it to the 3rd column, so that the first two rows of A now become either $\begin{pmatrix}1\,1\,0\dots0\\1\,1\,0\dots0\end{pmatrix}$ or $\begin{pmatrix}1\,1\,0\dots0\\0\,1\,1\dots0\end{pmatrix}$. In the second case, we continue the process (find the row that has 1 at the 3rd column, etc.), until, for some $n_1 \le 2010$, we meet the row that has 1 at the first column; the first n_1 rows of A now become $\begin{pmatrix}1\,1\,0\dots0\,0\,0\dots0\\0\,1\,1\dots0\,0\,0\dots0\\\vdots\,\vdots\,\vdots\dots\vdots\,\vdots\,\vdots\dots\vdots\\0\,0\,0\dots1\,1\,0\dots0\\1\,0\,0\dots0\,1\,0\dots0\end{pmatrix}$.

We then pass to the rows and columns of A from (n_1+1)st to 2010th, and repeat the procedure. After m such steps, we reduce A to the form $\begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & A_m \end{pmatrix}$, where for each j, A_j is an $n_j \times n_j$ matrix of the form $\begin{pmatrix} 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 1 \\ 1 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$.

We now have $n_1 + \ldots + n_m = 2010$ and $\det A = \prod_{j=1}^m \det A_j$. The determinant of each block A_j equals 1 ± 1: it is 0 if n_j is even, and 2 if n_j is odd. Thus, if n_j is even for some j, then $\det A = 0$; if all n_j are odd, then $\det A = \pm 2^m$, and in this case, since 2010 is even, m is even.

5. *Evaluate* $\lim\limits_{n\to\infty} n \sin(2\pi n! e)$.

<u>*Solution*</u>. Everyone knows that $e = \sum_{k=0}^{\infty} \frac{1}{k!}$. Thus, for any $n \in \mathbb{N}$, we have $n!e = m_n + t_n$, where $m_n = \sum_{k=0}^{n} \frac{n!}{k!}$ is an integer and $t_n = \sum_{k=n+1}^{\infty} \frac{n!}{k!}$. Since sin is a 2π-periodic function, for any $n \in \mathbb{N}$ we get $\sin(2\pi n!e) = \sin(2\pi m_n + 2\pi t_n) = \sin(2\pi t_n)$.

Next, for any $k \geq n+1$, $\frac{n!}{k!} = \frac{1}{(n+1)(n+2)\dots k} < \frac{1}{(n+1)^{k-n}}$, so

$$\frac{1}{n+1} < t_n < \sum_{k=n+1}^{\infty} \frac{1}{(n+1)^{k-n}} = \frac{1}{n+1} \cdot \frac{1}{1 - \frac{1}{n+1}} = \frac{1}{n}$$

Since both $n \sin\left(\frac{2\pi}{n+1}\right) = \frac{\sin\left(\frac{2\pi}{n+1}\right)}{1/n} \longrightarrow 2\pi$ and $n \sin\left(\frac{2\pi}{n}\right) = \frac{\sin\left(\frac{2\pi}{n}\right)}{1/n} \longrightarrow 2\pi$ as $n \to \infty$, by the squeeze theorem $\lim_{n\to\infty} n \sin(2\pi t_n) = 2\pi$, and so $\lim_{n\to\infty} n \sin(2\pi n! e) = 2\pi$.

6. *Let* α *be a real number. Find* $\lim\limits_{n\to\infty} \begin{pmatrix} 1 & \alpha/n \\ -\alpha/n & 1 \end{pmatrix}^n$.

<u>*Solution*</u>. It is well known(!) that the ring of 2×2 real matrices of the form $\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$ is isomorphic to the field of complex numbers, where the isomorphism is given by the formula $\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right) \leftrightarrow a + bi \in \mathbb{C}$ and is a mapping continuous in both directions. ($\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$ is the matrix of the linear transformation $z \mapsto (a+bi)z$ of $\mathbb{C} = \mathbb{R}^2$.) Since $\lim_{n\to\infty} \left(1 + \frac{i\alpha}{n}\right)^n = e^{i\alpha} = \cos\alpha + i \sin\alpha$, we obtain $\lim\limits_{n\to\infty} \left(\begin{smallmatrix} 1 & \alpha/n \\ -\alpha/n & 1 \end{smallmatrix}\right)^n = \left(\begin{smallmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{smallmatrix}\right)$.

<u>*Another solution*</u>. For any n, $\left(\begin{smallmatrix} 1 & \alpha/n \\ -\alpha/n & 1 \end{smallmatrix}\right) = r_n \left(\begin{smallmatrix} \cos\alpha_n & \sin\alpha_n \\ -\sin\alpha_n & \cos\alpha_n \end{smallmatrix}\right)$, where $r_n = \sqrt{1 + (\frac{\alpha}{n})^2}$ and $\alpha_n = \arctan(\alpha/n)$, $n \in \mathbb{N}$; thus $\left(\begin{smallmatrix} 1 & \alpha/n \\ -\alpha/n & 1 \end{smallmatrix}\right)^n = r_n^n \left(\begin{smallmatrix} \cos n\alpha_n & \sin n\alpha_n \\ -\sin n\alpha_n & \cos n\alpha_n \end{smallmatrix}\right)$. Since $r_n^n = \sqrt{\left(1 + \frac{\alpha^2}{n^2}\right)^n} \longrightarrow 1$ and $n\alpha_n = n \arctan(\alpha/n) \to \alpha$ as $n \to \infty$, we get $\lim\limits_{n\to\infty} \left(\begin{smallmatrix} 1 & \alpha/n \\ -\alpha/n & 1 \end{smallmatrix}\right)^n = \left(\begin{smallmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{smallmatrix}\right)$.

<u>*Yet another solution*</u>. For any $n \in \mathbb{N}$, the matrix $R_n = \left(\begin{smallmatrix} 1 & \alpha/n \\ -\alpha/n & 1 \end{smallmatrix}\right)$ has eigenvalues $1 + \frac{\alpha}{n} i$ and $1 - \frac{\alpha}{n} i$ (these are the roots of the polynomial $(1-x)^2 + \frac{\alpha^2}{n^2}$), and the corresponding eigenvector are $\left(\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right)$. So $R_n = P \left(\begin{smallmatrix} 1+\frac{\alpha}{n}i & 0 \\ 0 & 1-\frac{\alpha}{n}i \end{smallmatrix}\right) P^{-1}$ where $P = \left(\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right)$. Hence,

$R_n^n = P\begin{pmatrix} 1+\frac{\alpha}{n}i & 0 \\ 0 & 1-\frac{\alpha}{n}i \end{pmatrix}^n P^{-1} = P\begin{pmatrix} (1+\frac{\alpha}{n}i)^n & 0 \\ 0 & (1-\frac{\alpha}{n}i)^n \end{pmatrix} P^{-1}$. Since $\lim_{n\to\infty}(1 \pm \frac{\alpha}{n}i)^n = e^{\pm i\alpha}$, we get $\lim_{n\to\infty} R_n^n = P\begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} P^{-1} = \frac{1}{2}\begin{pmatrix} e^{i\alpha}+e^{-i\alpha} & -ie^{i\alpha}+ie^{-i\alpha} \\ ie^{i\alpha}-ie^{-i\alpha} & e^{i\alpha}+e^{-i\alpha} \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$.

And one more solution. Observe that for the matrix $A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$ one has $A^2 = \begin{pmatrix} -a^2 & 0 \\ 0 & -a^2 \end{pmatrix}$, $A^3 = \begin{pmatrix} 0 & -a^3 \\ a^3 & 0 \end{pmatrix}$, etc. Thus, for any n,

$$\begin{pmatrix} 1 & \alpha/n \\ -\alpha/n & 1 \end{pmatrix}^n = \left(I + \begin{pmatrix} 0 & \alpha/n \\ -\alpha/n & 0 \end{pmatrix}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \begin{pmatrix} 0 & \alpha/n \\ -\alpha/n & 0 \end{pmatrix}^k$$

$$= \begin{pmatrix} 1-\binom{n}{2}\frac{\alpha^2}{n^2}+\binom{n}{4}\frac{\alpha^4}{n^4}+\dots & n\frac{\alpha}{n}-\binom{n}{3}\frac{\alpha^3}{n^3}+\binom{n}{5}\frac{\alpha^5}{n^5}\dots \\ -n\frac{\alpha}{n}+\binom{n}{3}\frac{\alpha^3}{n^3}-\binom{n}{5}\frac{\alpha^5}{n^5}\dots & 1-\binom{n}{2}\frac{\alpha^2}{n^2}+\binom{n}{4}\frac{\alpha^4}{n^4}+\dots \end{pmatrix}$$

It remains to show that $\lim_{n\to\infty}\sum_{k=0}^{[n/2]}(-1)^k\binom{n}{2k}\frac{\alpha^{2k}}{n^{2k}} = \sum_{k=0}^{\infty}(-1)^k\frac{\alpha^{2k}}{(2k)!} = \cos\alpha$ and $\lim_{n\to\infty}\sum_{k=0}^{[(n-1)/2]}(-1)^k\binom{n}{2k}\frac{\alpha^{2k}}{n^{2k}} = \sum_{k=0}^{\infty}(-1)^k\frac{\alpha^{2k+1}}{(2k+1)!} = \sin\alpha$.

We will prove this for the cos function only, the proof for sin is similar. Observe that for any $k \in \mathbb{N}$, $\binom{n}{k}\frac{\alpha^k}{n^k} = \frac{n(n-1)\dots(n-k+1)}{n^k}\frac{\alpha^k}{k!} \longrightarrow \frac{\alpha^k}{k!}$ as $n \to \infty$. Let $\varepsilon > 0$. The series $\sum_{k=0}^{\infty}\frac{|\alpha|^{2k}}{(2k)!}$ converges, thus there exists N such that $\sum_{k=[N/2]+1}^{\infty}\frac{|\alpha|^{2k}}{(2k)!} < \varepsilon$. Then also $\left|\sum_{k=[N/2]+1}^{\infty}(-1)^k\frac{\alpha^{2k}}{(2k)!}\right| < \varepsilon$, and for any $n > N$,

$$\Big|\sum_{k=[N/2]+1}^{[n/2]}(-1)^k\binom{n}{2k}\frac{\alpha^{2k}}{n^{2k}}\Big| < \varepsilon.$$

Since $\binom{n}{2k}\frac{\alpha^{2k}}{n^{2k}} \longrightarrow \frac{\alpha^{2k}}{(2k)!}$ for $k = 0, \dots, [N/2]$, if $n > N$ is large enough we also have

$$\Big|\sum_{k=0}^{[N/2]}(-1)^k\binom{n}{2k}\frac{\alpha^{2k}}{n^{2k}} - \sum_{k=0}^{[N/2]}(-1)^k\frac{\alpha^{2k}}{(2k)!}\Big| < \varepsilon.$$

Hence, for such n,

$$\Big|\sum_{k=0}^{[n/2]}(-1)^k\binom{n}{2k}\frac{\alpha^{2k}}{n^{2k}} - \sum_{k=0}^{\infty}(-1)^k\frac{\alpha^{2k}}{(2k)!}\Big| < 3\varepsilon.$$