Solutions to 2011 Gordon Prize examination problems

1. *Let $1 \le a_1, \dots, a_{1006} \le 2011$ be distinct positive integers. Prove that there exist i, j such that $a_i + a_j = 2012$.*

Solution. Of course, there is nothing special in 2011, except that this is the current year A.D.; we will prove that for any $n \in \mathbb{N}$, if A is an n-element subset of $\{1, \dots, 2n-1\}$ then there are $a, b \in A$ with $a + b = 2n$. Indeed, let $B = 2n - A = \{2n - a,\ a \in A\}$. Then $B \subset \{1, \dots, 2n-1\}$ and $|B| = |A| = n$, so $A \cap B \ne \emptyset$. Let $a \in A \cap B$; since $a \in B$, we have $b = 2n - a \in A$, and so, $a, b \in A$, $a + b = 2n$.

2. *Let $g(x) = a_0 x^{r_0} + a_1 x^{r_1} + a_2 x^{r_2} + \dots + a_{2011} x^{r_{2011}}$, $x > 0$, where $a_0, \dots, a_{2011}$ are nonzero real numbers and $r_0, \dots, r_{2011}$ are distinct real numbers. Prove that g has at most 2011 zeroes in $(0, \infty)$.*

Solution. Again, 2011 appears here for fun only; we will prove by induction that for any $n \ge 0$, nonzero real numbers $a_0, \dots, a_n$ and distinct real numbers $r_0, \dots, r_n$ the function $g(x) = a_0 x^{r_0} + a_1 x^{r_1} + a_2 x^{r_2} + \dots + a_n x^{r_n}$ has at most n zeroes in $(0, \infty)$. For $n = 0$ this statement is trivial. Let $f(x) = x^{-r_0} g(x) = a_0 + a_1 x^{s_1} + a_2 x^{s_2} + \dots + a_n x^{s_n}$, where $s_i = r_i - r_0$, $i = 1, \dots, n$; then f has the same zeroes in $(0, \infty)$ as g. We have $f'(x) = a_1 s_1 x^{s_1 - 1} + a_2 s_2 x^{s_2 - 1} + \dots + a_n s_n x^{s_n - 1}$, which, after collecting simular terms, takes the form $f'(x) = b_0 x^{p_0} + b_1 x^{p_1} + \dots + b_m x^{p_m}$ for some $m < n$, nonzero b_i and distinct p_i. (If $f' = 0$, then f is constant, and so, has no roots at all (f cannot be equal to zero identically since a_i are nonzero); thus, we may assume that the expression for f' is nontrivial.) By (complete) induction, f' has at most m zeroes. But then f has at most $m + 1 \le n$ zeroes, since, by Rolle's theorem, between any two roots of f there is a root of f'.

3. *Prove that the "Pascal matrix"* $P_n = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 3 & \dots & n \\ 1 & 3 & 6 & \dots & \binom{n+1}{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & n & \binom{n+1}{2} & \dots & \binom{2n-2}{n-1} \end{pmatrix}$ *has determinant 1.*

Solution. The (i,j)th element of P_n is $\binom{i+j-2}{j-1}$. Let us subtract from each row of P_n, starting from the last one and excepting the first one, the preceding row; then the (i,j)th element, with $i \ge 2$, of the obtained matrix P_n' is $\binom{i+j-2}{j-1} - \binom{i+j-3}{j-1} = \binom{i+j-3}{j-2}$ (we use the identity $\binom{m}{k} = \binom{m-1}{k} + \binom{m-1}{k-1}$):

$$P_n' = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 2 & \dots & n-1 \\ 0 & 1 & 3 & \dots & \binom{n}{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 1 & n & \dots & \binom{2n-3}{n-2} \end{pmatrix}$$

Now subtract from each column of P_n' starting from the last one and excepting the first one, the preceding column; then the (i,j)th element, with $i, j \ge 2$, of the obtained matrix P_n'' is $\binom{i+j-3}{j-2} - \binom{i+j-4}{j-3} = \binom{i+j-4}{j-2}$:

$$P_n'' = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & 1 & 2 & \dots & n-1 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 1 & n-1 & \dots & \binom{2n-4}{n-2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & P_{n-1} \end{pmatrix},$$

where P_{n-1} the the "Pascal matrix" of size $(n-1) \times (n-1)$. By induction on n, $\det P_{n-1} = 1$, and so, $\det P_n'' = 1$. Since the row-column operations do not affect the determinant of a matrix, we obtain that

$\det P_n = \det P_n'' = 1$.

Another solution. We claim that $P_n = L_n U_n$, where

$$L_n = \begin{pmatrix} \binom{0}{0} & 0 & 0 & \dots & 0 \\ \binom{1}{0} & \binom{1}{1} & 0 & \dots & 0 \\ \binom{2}{0} & \binom{2}{1} & \binom{2}{2} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \binom{n-1}{0} & \binom{n-1}{1} & \binom{n-1}{2} & \dots & \binom{n-1}{n-1} \end{pmatrix} \text{ and } U_n = \begin{pmatrix} \binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \dots & \binom{n-1}{0} \\ 0 & \binom{1}{1} & \binom{2}{1} & \dots & \binom{n-1}{1} \\ 0 & 0 & \binom{2}{2} & \dots & \binom{n-1}{2} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & \binom{n-1}{n-1} \end{pmatrix} = L_n^t;$$

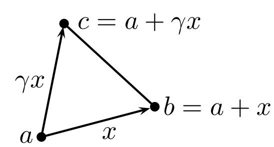
since $\det L_n = \det U_n = 1$, this implies that $\det P_n = 1$.

Indeed, if we assume that $\binom{i}{k} = 0$ for $k > i$, then the (i,j)-th entry of L_n is $\binom{i-1}{j-1}$ and the (i,j)-th entry of U_n is $\binom{j-1}{i-1}$, $i = 1, \dots, n$, $j = 1, \dots, n$. Hence, the (i,j)-th entry of $L_n U_n$ is equal to $\sum_{k=1}^{n} \binom{i-1}{k-1}\binom{j-1}{k-1}$, which is (well known to be) equal to $\binom{i+j-2}{j-1}$, the (i,j)-th entry of P_n.

4. *Let a, b, c be distinct complex numbers. Prove that the triangle $\triangle abc$ is equilateral iff*

$$a^2 + b^2 + c^2 = ab + bc + ca \qquad (*)$$

Solution. The triangle $\triangle abc$ is equilateral iff its side $c-a$ is obtained from its side $b-a$ by the rotation by $\pm\pi/3$, that is, if $c - a = \gamma(b-a)$ where $\gamma = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$:



We will now show that the equation $(*)$ is also equivalent to $c - a = \gamma(b-a)$. Put $x = b - a$ and $\alpha = (c-a)/(b-a)$, so that $b = a + x$ and $c = a + \alpha x$; then the equation $(*)$ is equivalent to

$$a^2 + (a^2 + 2ax + x^2) + (a^2 + 2a\alpha x + \alpha^2 x^2) = (a^2 + ax) + (a^2 + a\alpha x + ax + \alpha x^2) + (a^2 + a\alpha x),$$

which after all the cancelations takes the form

$$\alpha^2 - \alpha + 1 = 0,$$

which holds iff $\alpha = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$.

Another solution. The equation $(*)$ is equivalent to

$$a^2 + b^2 + b^2 + c^2 + c^2 + a^2 = 2ab + 2bc + 2ca,$$

that is, to

$$(a-b)^2 + (b-c)^2 + (c-a)^2 = 0.$$

This equation is, clearly, invariant under shifts (when a, b, c are replaced by $a+x, b+x, c+x$ respectively, with $x \in \mathbb{C}$) and under dilations-rotations (when a, b, c are replaced by $\alpha a, \alpha b, \alpha c$ respectively, with $\alpha \in \mathbb{C} \setminus \{0\}$). The property "the triangle $\triangle abc$ is equilateral" is also invariant under shifts and dilations-rotations; so, after an appropriate shift and dilation-rotation, we may assume that $a = 0$ and $b = 1$. In this case $(*)$ takes the form $c^2 - c + 1 = 0$, which means that $c = \frac{1}{2} \pm \frac{\sqrt{3}}{2}$; but this is equivalent to saying that the triangle $\triangle(0, 1, c)$ is equilateral.

5. *Prove that* $\displaystyle\int_{-2011}^{2011}\frac{dx}{1+x^{2011}+\sqrt{1+x^{4022}}}=2011.$

<u>*Solution*</u>. 2011 is irrelevant here as well... Let us show that for any $a>0$ and any continuous (or just integrable) odd function f on $[-a,a]$,

$$\int_{-a}^{a}\frac{dx}{1+f(x)+\sqrt{1+f(x)^2}}=a.$$

(Notice that $\sqrt{1+f(x)^2}>|f(x)|$, so $1+f(x)+\sqrt{1+f(x)^2}>0$ for all x.) Indeed, let $g(x)=\frac{1}{1+f(x)+\sqrt{1+f(x)^2}}$; then

$$g(x)+g(-x)=\frac{1}{1+f(x)+\sqrt{1+f(x)^2}}+\frac{1}{1+f(-x)+\sqrt{1+f(-x)^2}}=\frac{1}{1+f(x)+\sqrt{1+f(x)^2}}+\frac{1}{1-f(x)+\sqrt{1+f(x)^2}}$$
$$=2\frac{1+\sqrt{1+f(x)^2}}{1+1+f(x)^2+2\sqrt{1+f(x)^2}-f(x)^2}=2\frac{1+\sqrt{1+f(x)^2}}{2+2\sqrt{1+f(x)^2}}=1.$$

But $\int_{-a}^{a}g(-x)dx=\int_{-a}^{a}g(x)dx$, so

$$\int_{-a}^{a}g(x)dx=\frac{1}{2}\Big(\int_{-a}^{a}g(x)dx+\int_{-a}^{a}g(-x)dx\Big)=\frac{1}{2}\int_{-a}^{a}\big(g(x)+g(-x)\big)dx=\frac{1}{2}\int_{-a}^{a}dx=a.$$

6. *Define* $F\colon\mathbb{C}^3\longrightarrow\mathbb{C}^3$ *by* $F(x,y,z)=(x+y+z,xy+yz+zx,xyz)$. *Prove that* F *is surjective.*

<u>*Solution*</u>. Let $(a,b,c)\in\mathbb{C}^3$; we need to find $x,y,z\in\mathbb{C}$ such that $F(x,y,z)=(a,b,c)$. Consider the polynomial $p(w)=w^3-aw^2+bw-c$. By the fundamental theorem of arithemtic, $p(w)=(w-x)(w-y)(w-z)$ for some $x,y,z\in\mathbb{C}$, and by Vieta's theorem (or simply by opening the parentheses), $x+y+z=a$, $xy+yz+zx=b$, and $xyz=c$.