Solutions to 2012 Gordon Prize examination problems

1. *Let $n \in \mathbb{N}$. Find all complex solutions of the system of equations*

$$\begin{cases} x_1 + \ldots + x_n = 0 \\ x_1^2 + \ldots + x_n^2 = 0 \\ \vdots \qquad\qquad\quad \vdots \\ x_1^n + \ldots + x_n^n = 0 \end{cases} \qquad (*)$$

<u>*Solution*</u>. This system has only zero solution, $x_1 = x_2 = \ldots = x_n = 0$, which we are going to prove. Let $(x_1, \ldots, x_n)$ be an arbitrary solution of the system $(*)$. Let $p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ be the polynomial whose roots are $x_1, \ldots, x_n$, that is, $p(x) = (x - x_1)\ldots(x - x_n)$. Adding the identities $x_1^n + a_{n-1}x_1^{n-1} + \ldots + a_1 x_1 + a_0 = 0$, $x_2^n + a_{n-1}x_2^{n-1} + \ldots + a_1 x_2 + a_0 = 0$, ..., $x_1^n + a_{n-1}x_n^{n-1} + \ldots + a_1 x_n + a_0 = 0$, we obtain

$$(x_1^n + \ldots + x_n^n) + a_{n-1}(x_1^{n-1} + \ldots + x_n^{n-1}) + \ldots + a_1(x_1 + \ldots + x_n) + na_0 = 0,$$

from which and $(*)$, $na_0 = 0$. But $a_0 = (-1)^n x_1 \ldots x_n$, so, one of $x_i = 0$. Assume without loss of generality that $x_n = 0$; then $(*)$ implies

$$\begin{cases} x_1 + \ldots + x_{n-1} = 0 \\ x_1^2 + \ldots + x_{n-1}^2 = 0 \\ \vdots \\ x_1^{n-1} + \ldots + x_{n-1}^{n-1} = 0, \end{cases}$$

and by induction on n, also $x_1 = \ldots = x_{n-1} = 0$.

2. *Which number is greater,* $\log(5/4)$ *or* $\arctan(1/2)$*?*

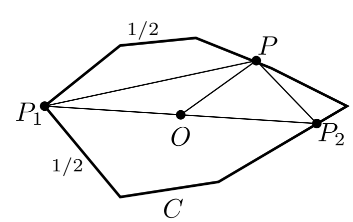
<u>*Solution*</u>. $\log(5/4) < \arctan(1/2)$. Indeed, $\log(5/4) = \int_0^{1/4} \frac{dx}{1+x}$ and $\arctan(1/2) = \int_0^{1/2} \frac{dx}{1+x^2}$; after the substitution $x = u^2$ we have

$$\int_0^{1/4} \frac{dx}{1+x} = \int_0^{1/2} \frac{2u\,du}{1+u^2} < \int_0^{1/2} \frac{du}{1+u^2}$$

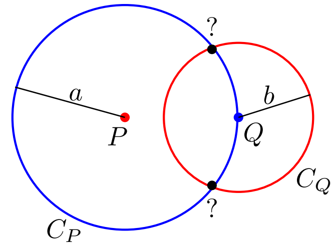
(since $2u < 1$ on $[0, 1/2)$).

3. *Prove that any closed polygonal (indeed, any rectifiable) curve C of length 1 in the plane is contained in a disk D of radius* $1/4$.

<u>*Solution*</u>. Choose two points P_1 and P_2 on C such that the length of both arcs of C connecting these points is $1/2$. Then for any point $P \in C$, $\operatorname{dist}(P_1, P)$ does not exceed the length of the arc of C connecting P_1 and P, and $\operatorname{dist}(P_2, P)$ does not exceed the length of the arc of C connecting P and P_2, so, $\operatorname{dist}(P_1, P) + \operatorname{dist}(P_2, P) \le 1/2$. Let O be the center of the interval $[P_1, P_2]$; then for any $P \in C$, $\operatorname{dist}(O, P) \le \frac{1}{2}\big(\operatorname{dist}(P_1, P) + \operatorname{dist}(P_2, P)\big) \le 1/4$. (The length of a median of a triangle never exceeds the half-sum of the lengths of the sides of the triangle passing from the same vertex.)



4. *Every point of the plane is colored one of two colors, red or blue. Let* $R = \{d(P,Q) : \text{both } P \text{ and } Q \text{ are red}\}$ *and* $B = \{d(P,Q) : \text{both } P \text{ and } Q \text{ are blue}\}$, *where* $d(P,Q)$ *denotes the distance between points* P *and* Q. *Prove that at least one of these sets* R, B *is equal to* $[0,\infty)$.



Solution. Assume that there exist positive numbers $a \notin R$ and $b \notin B$; this means that for any red point P all points in the plane at the distance of a from P are blue, and for any blue point Q all points at the distance of b from Q are red. Assume, without loss of generality, that $b \le a$. Choose a red point P, and let C_P be the circle of radius a centered at P; then all points of C_P are blue. Let Q be a point of C_P, and let C_Q be the circle of radius b centered at Q; then all points of C_Q are red. But the intersection of C_P and C_Q is nonempty, contradiction.

5. *Let* A *and* B *be two* $n \times n$ *matrices such that* $A + B = AB$. *Prove that* $AB = BA$.

Solution. We have $AB - A - B + I = I$, so $(A - I)(B - I) = I$, so $B - I = (A - I)^{-1}$, so $A - I$ and $B - I$ commute, so A and B commute.

6. *Let* $z_1, \ldots, z_n \in \mathbb{C}$ *and* $|z_1| = |z_2| = \ldots = |z_n| = r > 0$. *Prove that the number* $a = (z_1 + z_2)(z_2 + z_3) \ldots (z_{n-1} + z_n)(z_n + z_1)(z_1 z_2 \ldots z_n)^{-1}$ *is real.*

Solution. We have

$$\bar{a} = \frac{(\bar{z}_1 + \bar{z}_2)(\bar{z}_2 + \bar{z}_3)\ldots(\bar{z}_{n-1} + \bar{z}_n)(\bar{z}_n + \bar{z}_1)}{\bar{z}_1 \bar{z}_2 \ldots \bar{z}_n} = \frac{(\frac{r^2}{z_1} + \frac{r^2}{z_2})(\frac{r^2}{z_2} + \frac{r^2}{z_3})\ldots(\frac{r^2}{z_{n-1}} + \frac{r^2}{z_n})(\frac{r^2}{z_n} + \frac{r^2}{z_1})}{\frac{r^2}{z_1}\frac{r^2}{z_2}\ldots\frac{r^2}{z_n}}$$

$$= \frac{r^{2n}(\frac{1}{z_1} + \frac{1}{z_2})(\frac{1}{z_2} + \frac{1}{z_3})\ldots(\frac{1}{z_{n-1}} + \frac{1}{z_n})(\frac{1}{z_n} + \frac{1}{z_1})}{r^{2n}\frac{1}{z_1}\frac{1}{z_2}\ldots\frac{1}{z_n}} = \frac{(\frac{z_1+z_2}{z_1 z_2})(\frac{z_2+z_3}{z_2 z_3})\ldots(\frac{z_{n-1}+z_n}{z_{n-1} z_n})(\frac{z_n+z_1}{z_n z_1})}{\frac{1}{z_1 z_2 \ldots z_n}}$$

$$= \frac{\frac{(z_1+z_2)(z_2+z_3)\ldots(z_{n-1}+z_n)(z_n+z_1)}{(z_1 z_2 \ldots z_n)^2}}{\frac{1}{z_1 z_2 \ldots z_n}} = \frac{(z_1 + z_2)(z_2 + z_3)\ldots(z_{n-1} + z_n)(z_n + z_1)}{z_1 z_2 \ldots z_n} = a,$$

so a is real.