2014 Gordon exam solutions

1. *Prove that there does not exist a prime integer of the form* $1001001\ldots1001$.

Solution. The number $1001001\ldots1001$ having n digits 1 is

$$1+1000+\ldots+1000^{n-1}=\frac{1000^n-1}{1000-1}=\frac{(10^n-1)(100^n+10^n+1)}{999}.$$

This number can be prime only if one of the factors in the numerator is canceled by the denominator, which is only possible if $n\le 3$. However, the integers 1, 1001, and 1001001 are not prime. (1001 is divisible by 7, and 1001001 is divisible by 3.)

2. *Let $n\in\mathbb{N}$ and suppose that S is an $(n+1)$-element subset of the set $\{1,2,\ldots,2n\}$. Prove that there are $a,b\in S$ (not necessarily distinct) such that the sum $a+b$ is also in S.*

Solution. Let $S=\{a_1,a_2,\ldots,a_{n+1}\}$, where $a_1<a_2<\ldots<a_{n+1}$. Put $b_k=a_k-a_1$, $k=2,\ldots,n+1$, then $1\le b_2<b_3<\ldots<b_{n+1}<2n$. The set S has cardinality $n+1$ and the set $P=\{b_2,\ldots,b_{n+1}\}$ has cardinality n, thus they cannot be disjoint, and there exist i,j such that $a_i-a_1=b_i=a_j$, that is, $a_1+a_j=a_i\in S$.

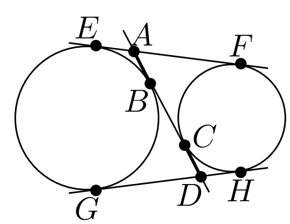
3. *Let $p(x)=x^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$ be a polynomial with complex coefficients satisfying $|a_i|\le 2014$, $i=0,\ldots,n-1$. If $z\in\mathbb{C}$ satisfies $p(z)=0$, prove that $|z|<2015$.*

Solution. Let $p(z)=0$, and let $|z|\ne 1$. Then

$$|z^n|=\big|a_{n-1}z^{n-1}+\ldots+a_1z+a_0\big|\le 2014\big(|z^{n-1}|+\ldots+|z|+1\big)=2014\frac{|z|^n-1}{|z|-1}.$$

If $|z|\ge 2015$, this implies that $|z|^n\le|z|^n-1$, contradiction.

4. *The straight lines on the picture are tangent to the circles. Prove that $|AB|=|CD|$.*



Solution. We have $|AB|=|AE|=|EF|-|AF|=|EF|-|AC|=|EF|-|AB|-|BC|$, so $|AB|=\frac{1}{2}(|EF|-|BC|)$. Similarly, $|CD|=\frac{1}{2}(|GH|-|BC|)$. Since $|EF|=|GH|$, we get the result.

5. *Suppose that all eigenvalues of an $n \times n$ matrix A are real and that* $\mathrm{tr}(A^2) = \mathrm{tr}(A^3) = \mathrm{tr}(A^4)$*. Prove that* $\mathrm{tr}(A^k) = \mathrm{tr}(A)$ *for all* $k \in \mathbb{N}$*.*

Solution. Let $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ be the eigenvalues of A, counting with multiplicity. Then for any k, $\mathrm{tr}\, A^k = \sum_{i=1}^n \lambda_i^k$. Since $\mathrm{tr}(A^2) = \mathrm{tr}(A^3) = \mathrm{tr}(A^4)$, we have $\mathrm{tr}(A^2) - 2\,\mathrm{tr}(A^3) + \mathrm{tr}(A^4) = 0$, so

$$0 = \sum_{i=1}^n \lambda_i^2 - 2\sum_{i=1}^n \lambda_i^3 + \sum_{i=1}^n \lambda_i^4 = \sum_{i=1}^n (\lambda_i^2 - 2\lambda_i^3 + \lambda_i^4) = \sum_{i=1}^n \lambda_i^2 (1-\lambda_i)^2,$$

which implies that for each i, $\lambda_i \in \{0,1\}$. Thus, for any k, $\sum_{i=1}^n \lambda_i^k = \sum_{i=1}^n \lambda_i = \mathrm{tr}\, A$.

6. *Prove that* $\displaystyle\int_0^{\pi/2} \log(\sin x)\, dx = -\frac{\pi}{2}\log 2.$

Solution.

$$\begin{aligned}
\int_0^{\pi/2} \log(\sin x)\, dx &= \int_0^{\pi/2} \log\big(2\sin(x/2)\cos(x/2)\big)\, dx \\
&= \int_0^{\pi/2} \big(\log 2 + \log(\sin(x/2)) + \log(\cos(x/2))\big)\, dx \\
&= \frac{\pi}{2}\log 2 + \int_0^{\pi/2} \log(\sin(x/2))\, dx + \int_0^{\pi/2} \log(\cos(x/2))\, dx \\
{\scriptstyle (y=x/2)} &= \frac{\pi}{2}\log 2 + 2\int_0^{\pi/4} \log(\sin y)\, dy + 2\int_0^{\pi/4} \log(\cos y)\, dy \\
&= \frac{\pi}{2}\log 2 + 2\int_0^{\pi/4} \log(\sin y)\, dy + 2\int_0^{\pi/4} \log(\sin(\pi/2 - y))\, dy \\
&= \frac{\pi}{2}\log 2 + 2\int_0^{\pi/4} \log(\sin y)\, dy + 2\int_{\pi/4}^{\pi/2} \log(\sin y)\, dy \\
&= \frac{\pi}{2}\log 2 + 2\int_0^{\pi/2} \log(\sin y)\, dy.
\end{aligned}$$

So, $\displaystyle\int_0^{\pi/2} \log(\sin x)\, dx = -\frac{\pi}{2}\log 2.$