2016 Gordon exam solutions

1. *Find all real x satisfying the equation*

$$\sqrt{\tfrac{x-2}{2018}}+\sqrt{\tfrac{x-3}{2017}}+\sqrt{\tfrac{x-4}{2016}}=\sqrt{\tfrac{x-2018}{2}}+\sqrt{\tfrac{x-2017}{3}}+\sqrt{\tfrac{x-2016}{4}}.$$

Solution. For $a>b>0$, consider the function $f(x)=\sqrt{\frac{x-a}{b}}-\sqrt{\frac{x-b}{a}}$, $x>a$. We have

$$f(x)=\frac{\frac{x-a}{b}-\frac{x-b}{a}}{\sqrt{\frac{x-a}{b}}+\sqrt{\frac{x-b}{a}}}=\frac{1}{ab}\cdot\frac{(a-b)x-(a^2-b^2)}{\sqrt{\frac{x-a}{b}}+\sqrt{\frac{x-b}{a}}}=\frac{a-b}{ab}\cdot\frac{x-(a+b)}{\sqrt{\frac{x-a}{b}}+\sqrt{\frac{x-b}{a}}},$$

so $f(x)=0$ for $x=a+b$, $f(x)>0$ for $x>a+b$, and $f(x)<0$ for $x<a+b$. Since $2018+2=2017+3=2016+4=2020$, it follows that the difference

$$\Big(\sqrt{\tfrac{x-2018}{2}}+\sqrt{\tfrac{x-2017}{3}}+\sqrt{\tfrac{x-2016}{4}}\Big)-\Big(\sqrt{\tfrac{x-2}{2018}}+\sqrt{\tfrac{x-3}{2017}}+\sqrt{\tfrac{x-4}{2016}}\Big)$$
$$=\Big(\sqrt{\tfrac{x-2018}{2}}-\sqrt{\tfrac{x-2}{2018}}\Big)+\Big(\sqrt{\tfrac{x-2017}{3}}-\sqrt{\tfrac{x-3}{2017}}\Big)+\Big(\sqrt{\tfrac{x-2016}{4}}-\sqrt{\tfrac{x-4}{2016}}\Big)$$

is positive for $x>2020$, negative for $x<2020$, and is equal to zero for $x=2020$ only.

2. *Suppose $z_1,\dots,z_k$ are complex numbers of absolute value 1; for each $n=1,2,\dots$ put $w_n=z_1^n+\dots+z_k^n$. If the sequence (w_n) converges, prove that $z_1=\dots=z_k=1$.*

Solution 1. For any $\varepsilon>0$ there are infinitely many n such that $|z_i^n-1|<\varepsilon$ for all $i=1,\dots,k$, so $|w_n-k|<k\varepsilon$ for such n. (This is a classical fact, but here is the proof: the sequence $v_n=(z_1^n,\dots,z_k^n)$ runs over the "torus" $S=\big\{(u_1,\dots,u_k)\in\mathbb{C}^k:|u_1|=\dots=|u_k|=1\big\}$. Since S is compact, given $\varepsilon>0$, there is an increasing sequence of indices $m_1<m_2<\cdots$ such that $|v_{m_j}-v_{m_1}|<\varepsilon$ for all $j\in\mathbb{N}$. But

$$|v_{m_j}-v_{m_1}|\ge|z_i^{m_j}-z_i^{m_1}|=|z_i^{m_1}|\cdot|z_i^{m_j-m_1}-1|=|z_i^{m_j-m_1}-1|$$

for all $i=1,\dots,k$, so $|z_i^{m_j-m_1}-1|<\varepsilon$, $i=1,\dots,k$, for all j.)

This implies that if the sequence (w_n) converges, then its limit must be equal to k. However if $z_i\ne1$ for some i, we have $\operatorname{Re}z_i^n<0$ for infinitely many n, and then $\operatorname{Re}w_n<k-1$ for such n.

Solution 2. By induction on k, we'll prove a more general statement: if $z_1,\dots,z_k$ are distinct complex numbers of absolute value 1, $\lambda_1,\dots,\lambda_k$ are nonzero complex numbers, and the sequence $w_n=\lambda_1z_1^n+\dots+\lambda_kz_k^n$, $n\in\mathbb{N}$, converges, then $k=1$ and $z_1=1$. The case $k=1$ is clear; assume that $k\ge2$. If one of z_i equals 1, we can exclude it; so, let's assume that $z_i\ne1$ for all i. The sequence

$$w_{n+1}-w_n=(z_1-1)\lambda_1z_1^n+\dots+(z_k-1)\lambda_1z_k^n,\ n\in\mathbb{N}$$

converges to 0, thus the sequence

$$(z_1-1)\lambda_1+(z_2-2)\lambda_2(z_2z_1^{-1})^n+\dots+(z_k-1)\lambda_1(z_kz_1^{-1})^n,\ n\in\mathbb{N}$$

also converges to 0, thus the sequence

$$(z_2-1)\lambda_2(z_2z_1^{-1})^n+\dots+(z_k-1)\lambda_1(z_kz_1^{-1})^n,\ n\in\mathbb{N}$$

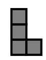
converges. By induction, $k-1=1$ and $z_2z_1^{-1}=1$, so $z_2=z_1$, contradiction.

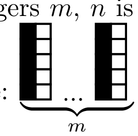
3. *In an invertible $n \times n$ matrix, what is the maximal number of entries that can be equal to 1?*

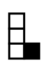
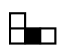
Solution. The answer is $n^2 - n + 1$. First of all, if a matrix has $\geq n^2 - n + 2$ entries equal to 1, then all entries in some two rows of the matrix are all equal to 1 and the matrix is degenerate. An example of an invertible matrix with $n^2 - n + 1$ entries equal to 1 is $\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{bmatrix}$. Indeed, after subtracting the first row from all other rows we obtain the matrix $\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & -1 \end{bmatrix}$, which is clearly nondegenerate.

4. *Suppose p is a polynomial with integer coefficients having at least 3 distinct integer roots. Prove that the equation $p(x) = 1$ has no integer solutions.*

Solution. If p is a polynomial with integer coefficients, then for any integer a, b one has $(b-a) \mid (p(b)-p(a))$. (Indeed, $(b-a) \mid (b^n - a^n)$ for any n.) Assume that $p(b) = 1$ for some $b \in \mathbb{Z}$ and let $a \in \mathbb{Z}$ be a root of p. Then $(b-a) \mid (p(b) - p(a)) = 1$, so $a = b \pm 1$; hence, p may have at most two integer roots.

5. *An L-tetromino is an L-shape made of four unit squares:* . *Suppose that an $m \times n$ chessboard is tiled by k L-tetrominos; prove that k is even.*

Solution. The total number of squares on the board is $m \times n = 4k$, so at least one of the integers m, n is even. Assume w.l.o.g. that m is even. Color the columns of the board alternatingly black-white: 

Now every tetromino covers either three black squares and one white square:  or , or one black square and three white squares:  or . Since the number of the black squares on the board equals the number of the white squares, there must be equal numbers of the "three-black-one-white" tetrominos and of the "one-black-three-white" tetrominos, and so, the total number of tetrominos is even.

6. *For a quadratic polynomial p define the quadratic polynomials T_1p and T_2p as follows:*

$$T_1p(x) = x^2 p\big(1 + \tfrac{1}{x}\big) \text{ and } T_2p(x) = (x-1)^2 p\big(\tfrac{1}{x-1}\big).$$

Applying the operations T_1 and T_2 in some order, is it possible to transform $x^2 + 1$ to $x^2 + 2017x + 1$?

Solution 1. Notice that T_2 is the inverse of T_1: $T_2T_1p(x) = p(x)$ for any quadratic polynomials p. Thus the composition of any finite sequence of the transformations T_1 and T_2 equals T_1^n for some $n \in \mathbb{Z}$. For any p, $T_1p(x) = x^2 p\big(\frac{x+1}{x}\big)$, $T_1^2p(x) = (x+1)^2 p\big(\frac{2x+1}{x+1}\big)$, $T_1^3p(x) = (2x+1)^2 p\big(\frac{3x+2}{2x+1}\big)$, etc., and by induction, for any $n \in \mathbb{N}$,

$$T_1^n p(x) = (F_n x + F_{n-1})^2 p\Big(\frac{F_{n+1}x + F_n}{F_n x + F_{n-1}}\Big),$$

where $F_0 = 0$, $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, ... is the Fibonacci sequence. The formula also works for $n \leq 0$ if we define $F_n = F_{n+2} - F_{n-1}$, so that $F_{-1} = 1$, $F_{-2} = -1$, $F_{-3} = 2$, ..., and $F_n = (-1)^n F_{-n}$ for any n. Thus for $p(x) = x^2 + 1$ and any $n \in \mathbb{Z}$ we have

$$T_1^n p(x) = (F_{n+1}x + F_n)^2 + (F_n x + F_{n-1})^2 = (F_{n+1}^2 + F_n^2)x^2 + 2(F_{n+1} + F_{n-1})F_n x + (F_n^2 + F_{n-1}^2),$$

and we never obtain $x^2 + 2017x + 1$.

Solution 2. We claim that both T_1 and T_2 preserve the discriminant of the polynomial they are applied to; thus, since the polynomials x^2+1 and $x^2+2017x+1$ have different discriminants, they cannot be transformed to each other. The claim can be checked by a direct computation, but the following approach works better. Let R and S be the operations on quadratic polynomials defined by $Rp(x) = x^2 p\big(\frac{1}{x}\big)$ and $Sp(x) = p(x+1)$;

then $T_1 = RS$ and $T_2 = S^{-1}R$. (Since $R^{-1} = R$, this implies by the way that $T_2 = T_1^{-1}$.) Clearly, R and S preserve the discriminant, so T_1 and T_2 also do.

Solution 3. For an arbitrary quadratic polynomial $p(x) = ax^2 + bx + c$ we have

$$T_1p(x) = x^2p\big(1+\tfrac{1}{x}\big) = x^2\Big(a\big(1+\tfrac{1}{x}\big)^2+b\big(1+\tfrac{1}{x}\big)+c\Big) = ax^2+2ax+a+bx^2+b+c = (a+b+c)x^2+(b+2a)x+c,$$

and

$$T_2p(x) = (x-1)^2p\big(\tfrac{1}{x-1}\big) = (x-1)^2\Big(a\big(\tfrac{1}{x-1}\big)^2+b\big(\tfrac{1}{x-1}\big)+c\Big) = a+bx-b+cx^2-2cx+c = cx^2+(b-2c)x+(a-b+c).$$

We see that both T_1 and T_2 preserve the parity of the coefficient of x in the polynomial. Since the polynomials $x^2+1 = x^2+0x+1$ and $x^2+2017x+1$ have different parities of the coefficient of x, they cannot be transformed to each other.

Solution 4. We see from the formulas produced for T_1 and T_2 in Solution 3 that $T_1(x^2+1) = 3x^2+2x+1$, and $T_2(x^2+1) = x^2-2x+2$. We will now prove by induction the following two claims:

Claim 1. *For all $n \geq 1$, the leading coefficient of $T_1^n(x^2+1)$ is larger than 1, the coefficient of x is positive, and the constant coefficient is 1.*

Proof. We proceed by induction. We see that the desired result holds for the base case of $n = 1$. Let us assume that the assertion holds for some $n \in \mathbb{N}$, that is $T_1^n(x^2+1) = ax^2+bx+1$ where $a > 1$ and $b > 0$. Then for $n+1$ we have

$$T_1^{n+1}(x^2+1) = T_1\big(T_1^n(x^2+1)\big) = T_1(ax^2+bx+c) = (a+b+c)x^2+(b+2a)x+c = (a+b+1)x^2+(b+2a)x+1,$$

with $a+b+1 > 1$, $b+2a > 0$, which implies the induction step. ■

Claim 2. *For all $n \geq 1$, the leading coefficient of $T_2^n(x^2+1)$ is at least 1, the coefficient of x is negative, and the constant coefficient is at least 1.*

Proof. We proceed by induction. We see that the desired result holds for the base case of $n = 1$. Assume that the assertion holds for some $n \in \mathbb{N}$, that is, $T_2^n(x^2+1) = ax^2+bx+c$ where $a \geq 1$, $b < 0$, and $c \geq 1$. Then

$$T_2^{n+1}(x^2+1) = T_2\big(T_2^n(x^2+1)\big) = T_2(ax^2+bx+c) = cx^2+(b-2c)x+(a-b+c),$$

with $c \geq 1$, $b-2c < 0$, and $a-b+c \geq 1$, which gives the induction step. ■

Returning to the main problem at hand, we see that $T_1^n(x^2+1) \neq x^2+2017x+1$ for any $n \geq 1$ since the leading coefficient of the left hand side will always be larger than 1. Similarly, $T_2^n(x^2+1) \neq x^2+2017x+1$ for any $n \geq 1$ since the coefficient of x on the left hand side will always be negative. And since the composition of any finite sequence of transformations T_1 and T_2, if nonidentical, equals either T_1^n or T_2^n for some $n \in \mathbb{N}$ (see Solution 1), we are done.

Solution 5. If a quadratic polynomial p has a real root $x_0 \neq 0$, then both T_1p and T_2p also have real roots, namely, $\frac{1}{x_0-1}$ and $\frac{1}{x_0}+1$ respectively. Now, the polynomial $x^2+2017x+1$ has a positive discriminant, hence two real roots, and x^2+1 has no real roots, thus they cannot be transformed to each other.

(In this solution we implicitely used the fact that $T_2 = T_1^{-1}$. If we prefer to avoid this, we should show that, conversely, if T_1p or T_2p have a real root, then p also does.)

Solution 6. We see from the formulas derived for T_1 and T_2 in Solution 3 that T_1 and T_2 act as linear transformations on the 3-dimensional $\mathbb{R}$-vector space of quadratic polynomials with real coefficients, whose matrices in the basis $\{1, x, x^2\}$ are

$$A_1 = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -1 & 1 \end{bmatrix} \text{ respectively.}$$

We now see that $A_1A_2 = I$, so $A_2 = A_1^{-1}$ (and so $T_2 = T_1^{-1}$). It follows that any sequence of applications of A_1 and A_2 to some initial vector v_0 reduces to $A_1^nv_0$ for some $n \in \mathbb{Z}$. Our goal is therefore to check whether there is an $n \in \mathbb{Z}$ such that

$$A_1^n\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2017 \\ 1 \end{bmatrix}. \qquad (*)$$

The characteristic polynomial of A_1 is $x^3 - 2x^2 - 2x + 1$, which factors to $(x+1)\big(x - \frac{3-\sqrt{5}}{2}\big)\big(x - \frac{3+\sqrt{5}}{2}\big)$. So, the matrix A_1 has three distinct eigenvalues, -1, $\frac{3-\sqrt{5}}{2}$ and $\frac{3+\sqrt{5}}{2}$, and hence, is diagonalizable. We then proceed to diagonalize the matrix to obtain

$$\begin{bmatrix}1&1&1\\2&1&0\\1&0&0\end{bmatrix} = \begin{bmatrix}-1 & \frac{1}{2}(3-\sqrt{5}) & \frac{1}{2}(3+\sqrt{5})\\ 1 & 1-\sqrt{5} & 1+\sqrt{5}\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}-1 & 0 & 0\\ 0 & \frac{1}{2}(3-\sqrt{5}) & 0\\ 0 & 0 & \frac{1}{2}(3+\sqrt{5})\end{bmatrix}\begin{bmatrix}-\frac{2}{5} & \frac{1}{5} & \frac{2}{5}\\ \frac{1}{5} & \frac{1}{10}(-1-\sqrt{5}) & \frac{1}{10}(3+\sqrt{5})\\ \frac{1}{5} & \frac{1}{10}(-1+\sqrt{5}) & \frac{1}{10}(3-\sqrt{5})\end{bmatrix}.$$

Thus equation $(*)$ reduces to

$$\begin{bmatrix}-1 & \frac{1}{2}(3-\sqrt{5}) & \frac{1}{2}(3+\sqrt{5})\\ 1 & 1-\sqrt{5} & 1+\sqrt{5}\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}(-1)^n & 0 & 0\\ 0 & (\frac{1}{2}(3-\sqrt{5}))^n & 0\\ 0 & 0 & (\frac{1}{2}(3+\sqrt{5}))^n\end{bmatrix}\begin{bmatrix}-\frac{2}{5} & \frac{1}{5} & \frac{2}{5}\\ \frac{1}{5} & \frac{1}{10}(-1-\sqrt{5}) & \frac{1}{10}(3+\sqrt{5})\\ \frac{1}{5} & \frac{1}{10}(-1+\sqrt{5}) & \frac{1}{10}(3-\sqrt{5})\end{bmatrix}\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}1\\2017\\1\end{bmatrix},$$

which, in turn, is equivalent to

$$\begin{bmatrix}(-1)^n & 0 & 0\\ 0 & (\frac{1}{2}(3-\sqrt{5}))^n & 0\\ 0 & 0 & (\frac{1}{2}(3+\sqrt{5}))^n\end{bmatrix}\begin{bmatrix}0\\ \frac{1}{10}(5+\sqrt{5})\\ \frac{1}{10}(5-\sqrt{5})\end{bmatrix} = \begin{bmatrix}\frac{2017}{5}\\ x\\ y\end{bmatrix}$$

for some real x and y, which we don't compute since it is already clear that this equation is not solvable for any integer n.

Remark. While this solution of the problem looks unreasonably long and computational, it however illustrates a general approach that applies to many similar problems.