2018 Gordon exam solutions

1. *Prove that for all $n \in \mathbb{N}$,* $2n \sqrt[2n]{\dfrac{n!}{(3n)!}} < \log 3$.

Solution. By the (generalized) arithmetic-geometric means inequality,

$$2n \sqrt[2n]{\frac{n!}{(3n)!}} = 2n \sqrt[2n]{\frac{1}{(n+1)(n+2)\cdots(3n)}} < 2n \cdot \frac{1}{2n}\Big(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n}\Big)$$
$$= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n}.$$

The sum

$$S = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n} = \frac{1}{n}\Big(\frac{1}{1+1/n} + \frac{1}{1+2/n} + \cdots + \frac{1}{1+2n/n}\Big)$$

is a lower Riemann sum of the function $\dfrac{1}{1+x}$ on the interval $[0, 2]$, so

$$S \le \int_0^2 \frac{1}{1+x}\,dx = \log 3 - \log 1 = \log 3.$$

2. *Determine whether the integer part of* $(1+\sqrt{2})^{2018}$ *is even or odd.*

Solution. For any $n \in \mathbb{N}$,

$$a_n = (1+\sqrt{2})^n + (1-\sqrt{2})^n = \textstyle\sum_{i=0}^{n} \binom{n}{i}\sqrt{2}^i + \sum_{i=0}^{n}(-1)^i\binom{n}{i}\sqrt{2}^i = 2\sum_{j=0}^{n/2}\binom{n}{2j}2^i,$$

which is an even integer. Since $1 < \sqrt{2} < 2$, we have $0 < (1-\sqrt{2})^n < 1$ if n is even and $-1 < (1-\sqrt{2})^n < 0$ if n is odd. So, the integer part of $(1+\sqrt{2})^n = a_n - (1-\sqrt{2})^n$ is odd if n is even, and is even if n is odd. In particular, the integer part of $(1+\sqrt{2})^{2018}$ is odd.

3. *Let $P = A_1A_2\ldots A_n$ be a regular n-gon with center O, and let $R = \mathrm{dist}(O, A_1)$. Prove that for any point X in the plane, $\sum_{k=1}^{n}\mathrm{dist}(X, A_k)^2 = n(R^2 + d^2)$, where $d = \mathrm{dist}(X, O)$.*

Solution. Let's identify the plane containing P with the complex plane $\mathbb{C}$ so that $O = 0$ and $A_{k+1} = R\omega^k$, $k = 0, \ldots, n-1$, where $\omega = e^{2\pi i/k}$. Then

$$\sum_{k=0}^{n-1}\mathrm{dist}(X, A_{k+1})^2 = \sum_{k=0}^{n-1}|X - R\omega^k|^2 = \sum_{k=0}^{n-1}(X - R\omega^k)(\overline{x} - R\overline{\omega}^k)$$
$$= \sum_{k=0}^{n-1}|X|^2 - \overline{x}R\sum_{k=0}^{n-1}\omega^k - XR\sum_{k=0}^{n-1}\overline{\omega}^k + \sum_{k=0}^{1}R^2|\omega|^k = n(|X|^2 + R^2).$$

(It is well known that $\sum_{k=0}^{n-1}\omega^k = (1-\omega^n)/(1-\omega) = 0$.)

4. *Suppose an ellipse E in the plane $\mathbb{R}^2$ has no points of the lattice $\mathbb{Z}^2$ in its interior. Prove that there are at most 4 points of $\mathbb{Z}^2$ on the boundary of E.*

Solution. Assume that there are ≥ 5 points of $\mathbb{Z}^2$ on the boundary of E. Then two of these points, $a = (n_1, n_2)$ and $b = (m_1, m_2)$, have the same parities of coordinates, so that $n_1 + m_1$ and $n_2 + m_2$ are both even. Then the midpoint $c = \frac{1}{2}(a+b) = \frac{1}{2}(n_1+m_1, n_2+m_2)$ of the interval (a, b) is a point of $\mathbb{Z}^2$, and since E is (strictly) convex, c is contained in the interior of E.

5. *Let T be a linear transformation of the vector space M_n of $n \times n$ (real) matrices such that $\det T(A) = \det A$ for all $A \in M_n$. Prove that T is invertible.*

Solution. In the way of contradiction, assume that $A \in \ker T$ and $A \neq 0$. Then for any $B \in M_n$,

$$\det(B + A) = \det(T(B + A)) = \det(T(B) + T(A)) = \det(T(A)) = \det(A).$$

The problem is solved once we prove the following:

Lemma. *For any nonzero $A \in M_n$, there exists $B \in M_n$ such that $\det(B) = 0$ and $\det(B + A) \neq 0$.*

Indeed, let $A = \big(u_1|u_2|\dots|u_n\big)$, where u_i are the columns of A, and, assume, without loss of generality, that $u_1 \neq 0$. Find a basis $\{u_1, v_2, \dots, v_n\}$ in $\mathbb{R}^n$ (or in F^n if our matrices are over a field F) whose first element is u_1. Now put $w_i = v_i - u_i$, $i = 2, \dots, n$, and $B = \big(0|w_2|\dots|w_n\big)$; then B is degenerate, but $B + A = \big(u_1|v_2|\dots|v_n\big)$ is not.

6. *Prove that $\sin 1^\circ$ is irrational.*

Solution. Assume that $\sin 1^\circ \in \mathbb{Q}$. Then also $\cos 2^\circ = 1 - 2\sin^2 1^\circ \in \mathbb{Q}$, and $\cos 4^\circ = 2\cos^2 2^\circ - 1 \in \mathbb{Q}$, amd by induction, $\cos 8^\circ, \cos 16^\circ, \cos 32^\circ \in \mathbb{Q}$. Now,

$$\cos 30^\circ = \cos 32^\circ \cos 2^\circ + \sin 32^\circ \sin 2^\circ.$$

We have that $\cos 32^\circ \cos 2^\circ \in \mathbb{Q}$, and also

$$\begin{aligned}\sin 32^\circ \sin 2^\circ &= 2\cos 16^\circ \sin 16^\circ \sin 2^\circ = 4\cos 16^\circ \cos 8^\circ \sin 8^\circ \sin 2^\circ = 8\cos 16^\circ \cos 8^\circ \cos 4^\circ \sin 4^\circ \sin 2^\circ \\ &= 16\cos 16^\circ \cos 8^\circ \cos 4^\circ \cos 2^\circ \sin^2 2^\circ = 8\cos 16^\circ \cos 8^\circ \cos 4^\circ \cos 2^\circ(1 - \cos 4^\circ) \in \mathbb{Q}.\end{aligned}$$

So, $\cos 30^\circ \in \mathbb{Q}$, which is false.

Second solution. If, for some x, $\sin x \in \mathbb{Q}$, then also $\cos 2x = 1 - 2\sin^2 x \in \mathbb{Q}$, and $\sin 3x = 3\sin x - 4\sin^3 x \in \mathbb{Q}$, and so $\sin 5x = 2\sin 3x \cos 2x - \sin x \in \mathbb{Q}$. Hence, if one had $\sin 1^\circ \in \mathbb{Q}$, then one would also have $\sin 3^\circ \in \mathbb{Q}$, so $\sin 9^\circ \in \mathbb{Q}$, so $\sin 45^\circ \in \mathbb{Q}$, which is not the case.

Third solution. For any $n \in \mathbb{N}$, $\cos(nx) = T_n(\cos x)$, where T_n is *the nth Chebyshev polynomial of the first kind*, which is a polynomial of degree n with integer coefficients. (The existence of these polynomials can be easily established by (complete) induction, using the formulas $\cos(2nx) = 2\cos^2(nx) - 1$ and $\cos((2n+1)x) = 2\cos((n+1)x)\cos(nx) - \cos x$.) If $\sin 1^\circ \in \mathbb{Q}$, then $\cos 2^\circ = 1 - 2\sin^2 1^\circ \in \mathbb{Q}$, and then $\cos 30^\circ = T_{15}(\cos 2^\circ) \in \mathbb{Q}$, which is not true.

Fourth solution. It also involves Chebyshev's polynomials. If $\sin 1^\circ$ is rational, then so is $\cos 89^\circ = \cos(90^\circ - 1^\circ) = \sin 1^\circ \in \mathbb{Q}$, and then rational is also $\cos(30 \cdot 89^\circ) = T_{30}(\cos 89^\circ)$. However,

$$\cos(30 \cdot 89^\circ) = \cos 30(90^\circ - 1^\circ) = \cos(30 \cdot 90^\circ - 30^\circ) = \cos(7 \cdot 360^\circ + 180^\circ - 30^\circ) = -\cos 30^\circ = -\sqrt{3}/2$$

is irrational.

Fifth solution. (For those who are familiar with the theory of field extensions.)
Assume that $\sin 1^\circ \in \mathbb{Q}$. Then $\cos 1^\circ = \sqrt{1 - \sin^2 1^\circ}$ is contained in an extension L of $\mathbb{Q}$ of degree at most 2. Then by induction, $\sin k^\circ = \sin(k-1)^\circ \cos 1^\circ + \sin 1^\circ \cos(k-1)^\circ$ and $\cos k^\circ = \cos(k-1)^\circ \cos 1^\circ - \sin 1^\circ \sin(k-1)^\circ$ are also contained in L for all integer k. However, no extension of $\mathbb{Q}$ of degree 2 contains both $\cos 30^\circ = \sqrt{3}/2$ and $\cos 45^\circ = \sqrt{2}/2$.