2019 Gordon exam solutions

1. *Prove that there are infinitely many primes p such that for some $n \in \mathbb{N}$ the integer n^2+n+1 is divisible by p.*

Solution. For any finite collection of prime integers $p_1, \ldots, p_k$ put $n = p_1 \cdots p_k$, then the integer n^2+n+1 is not divisible by any of p_i; it therefore has a prime divisor distinct from each of p_i.

2. *Let f be the function $(0,\infty) \longrightarrow \mathbb{R}$ defined by: $f(x) = 0$ if $x \notin \mathbb{Q}$, and $f(x) = 1/n^3$ if $x = m/n$ is rational in lowest terms. If $k \in \mathbb{N}$ is not a perfect square, prove that f is differentiable at $\sqrt{k}$.*

Solution. We have $f(\sqrt{k}) = 0$, and $f(x) = 0$ for all irrational x; to prove that f is differentiable at the point $\sqrt{k}$ with $f'(\sqrt{k}) = 0$ we only need to show that $\lim_{m/n \to \sqrt{k}} \frac{1/n^3}{|m/n-\sqrt{k}|} = 0$. For any $m, n \in \mathbb{N}$ such that $m/n < \sqrt{k}+1$ we have

$$\frac{1/n^3}{|m/n-\sqrt{k}|} = \frac{1}{n^3} \cdot \frac{m/n+\sqrt{k}}{|(m/n)^2-k|} < \frac{1}{n} \cdot \frac{2\sqrt{k}+1}{|m^2-kn^2|} < (2\sqrt{k}+1)/n$$

since $m^2 - kn^2$ is a nonzero integer. Since $n \longrightarrow \infty$ as $m/n \longrightarrow \sqrt{k}$, we have $\frac{1/n^3}{|m/n-\sqrt{k}|} \longrightarrow 0$.

3. *Find the maximum of the integral $\int_0^1 \big(x^{2020} f(x) - x^{2019} f(x)^2\big)\,dx$ over all continuous functions $f\colon [0,1] \longrightarrow \mathbb{R}$.*

Solution. For every $x \in [0,1]$, the expression $xf(x) - f(x)^2 = (x - f(x))f(x)$ maximizes when $f(x) = x/2$, in which case it equals $x^2/4$. Hence, the maximum value of

$$\int_0^1 \big(x^{2020} f(x) - x^{2019} f(x)^2\big)\,dx = \int_0^1 x^{2019}\big(xf(x) - f(x)^2\big)\,dx$$

is $\int_0^1 x^{2019}(x^2/4)\,dx = \frac{1}{8088}$, and is reached for $f(x) = x/2$, $x \in [0,1]$.

4. *Let $S = \{z \in \mathbb{C} : |z| = 1\}$. Suppose $z_1, \ldots, z_n \in S$ satisfy $\big|(z-z_1)\cdots(z-z_n)\big| \le 2$ for every $z \in S$. Prove that $z_1, \ldots, z_n$ are the vertices of a regular n-gon.*

Solution. Let $p(z) = (z-z_1)\cdots(z-z_n) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$. where $a_0 = \pm z_1 \cdots z_n$, and so, $|a_0| = 1$; after multiplying all z_i by $a_0^{1/n}$ we may and will assume that $a_0 = 1$. Let $\omega = e^{2\pi i/n}$; for each $1 \le d \le n-1$ we have

$$\sum_{k=0}^{n-1} (\omega^k)^d = \frac{1-\omega^{dn}}{1-\omega^d} = 0.$$

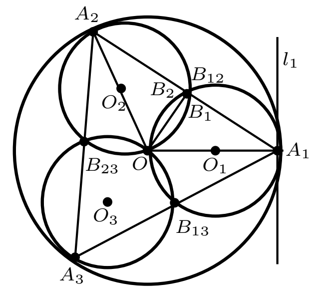
So,

$$\Big|\sum_{k=0}^{n-1} p(\omega^k)\Big| = \Big|\sum_{k=0}^{n-1}(\omega^k)^n + a_{n-1}\sum_{k=0}^{n-1}(\omega^k)^{n-1} + \cdots + a_1\sum_{k=0}^{n-1}(\omega^k)^1 + \sum_{k=0}^{n-1} a_0\Big| = |n+0+\cdots+0+n| = 2n.$$

Since $|p(\omega^k)| \le 2$ for all k, this implies that $|p(\omega^k)| = 2$ for all k. Hence, $1, \omega, \ldots, \omega^{n-1}$ are roots of the polynomial $p(z) - 2$, so, $p(z) - 2 = \prod_{k=0}^{n-1}(z - \omega_k) = z^n - 1$, and so $p(z) = z^n + 1$. Hence, the roots $z_1, \ldots, z_n$ of p are the numbers $e^{2\pi i(k/n+1/2)}$, which are indeed located at the vertices of a regular n-gon.

5. *Suppose C_1, C_2 and C_3 are cirlces of equal radius inscribed in a circle C and having a common intersection point O. For every $1 \le i \le 3$ let A_i be the tangency point of C_i and C, and for every $1 \le i < j \le 3$ let B_{ij} be the intersection point of C_i and C_j other than O. Prove that for each $1 \le i < j \le 3$, the points A_i, B_{ij}, and A_j are collinear.*

Solution. Let O_1, O_2 and O_3 be the centers of the circles C_1, C_2 and C_3 respectively, and let r be their radius. We claim that the point O is the center of circle C, and that OA_i are diameters of the circles C_i, $i = 1, 2, 3$. Indeed, let O' be the center and R be the radius of C. For every $i = 1, 2, 3$, the circles C_i and C have a common tangent line l_i at the point A_i, so the line orthogonal to l_i at A_i passes through both O_i and O', and so, $|O'O_i| = R - r$. Hence, O' is equidistant from the points O_1, O_2, O_3; but the point O is also equidistant from these points, with $|OO_i| = r$ for all i; thus $O' = O$, $R - r = r$, and OA_i are diameters of C_i.



Now, if B_1 is the point of intersection of the line A_1A_2 with C_1, then, since OA_1 is a diameter of C_1, we have $\angle OB_1A_1 = \pi/2$, so OB_1 is orthogonal to A_1A_2. But also OB_2 is orthogonal to A_1A_2, where B_2 is the point of intersection A_1A_2 with C_2; hence, $B_1 = B_2 = B_{1,2}$.

6. *Let* $A = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{pmatrix}$ *be an* $n \times n$ *real matrix with zero trace, i.e.* $\sum_{i=1}^{n} a_{i,i} = 0$. *Prove that* A *is conjugate to a matrix with zero main diagonal. (That is, prove there exists an invertible* $n \times n$ *matrix* P *such that* $PAP^{-1} = \begin{pmatrix} 0 & b_{1,2} & \dots & b_{1,n} \\ b_{2,1} & 0 & \dots & b_{2,n} \\ \vdots & \vdots & & \vdots \\ b_{n,1} & b_{n,2} & \dots & 0 \end{pmatrix}$ *for some real numbers* $b_{i,j}$.*)*

Solution. We will use induction on n; if $n = 1$, then $A = 0$. Assume that $A \neq 0$. Since A is not a scalar matrix, there exists $u \in \mathbb{R}^n$ such that Au and u are linearly independent; find a basis in $\mathbb{R}^n$ whose first two elements are u and Au. In this new basis (and the operation of change of basis is known to be equivalent to the operation of conjugation), A takes the form $A' = \begin{pmatrix} 0 & c_{1,2} & \dots & c_{1,n} \\ 1 & c_{2,2} & \dots & c_{2,n} \\ \vdots & \vdots & & \vdots \\ 0 & c_{n,2} & \dots & c_{n,n} \end{pmatrix} = \begin{pmatrix} 0 & C_1 \\ C_2 & C \end{pmatrix}$, where C_1, C_2 and C are, respectively, $1 \times (n-1)$, $(n-1) \times 1$ and $(n-1) \times (n-1)$ matrices, and $\operatorname{trace} C = \operatorname{trace} A' - 0 = \operatorname{trace} A - 0 = 0$. By induction, C is conjugate to a matrix QCQ^{-1} with zero main diagonal; then $\begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} A' \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix}^{-1} = \begin{pmatrix} 0 & C_1Q^{-1} \\ QC_2 & QCQ^{-1} \end{pmatrix}$ is a matrix with zero main diagonal which is conjugate to A.