Solutions for the Rasor-Bareis Prize Examination

1. There are exactly two such solutions. Note that: $e^{3\pi/2} > 2^{9/2} = 16\sqrt{2} > 20$; e^x is strictly increasing; and, $20\cos x \le 20$ for all $x \in \mathbb{R}$. Thus, there are no solutions for $x \ge \frac{3\pi}{2}$. Moreover, $\cos x \le 0$ for $x \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, so there are no solutions in $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$. Hence, it suffices to show that there are exactly two solutions in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Put $f(x) = e^x - 20\cos x$ for $x \in \mathbb{R}$. We have $f\left(-\frac{\pi}{2}\right) = e^{-\pi/2} > 0$, $f(0) = -19 < 0$, and $f\left(\frac{\pi}{2}\right) = e^{\pi/2} > 0$. Hence, by the Intermediate Value Theorem, there are at least two zeros of f in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. If there were more than two then by two applications of Rolle's Theorem there would have to be a zero of $f''(x) = e^x + 20\cos x$ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, a contradiction (since $\cos x > 0$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$).

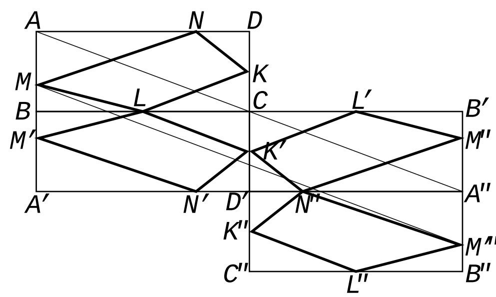
2. Consider the 2×2001 array

$$\begin{array}{cccccccc} 1 & 2 & \cdots & 1001 & 1002 & 1003 & \cdots & 2001 \\ 1001 & 1002 & \cdots & 2001 & 1 & 2 & \cdots & 1000 \end{array}$$

Every number from $\{1, \ldots, 2001\}$ appears exactly twice. Hence, there are exactly 2002 occurrences of selected numbers in the array, and so some two such occurrences appear in the same column, making the difference between some two selected numbers differ by 1000 or 1001.

3. Define $a_n = \underbrace{11\ldots1}_{n}$. Among infinitely many numbers $a_1, a_2, \ldots$ there are two, a_m and a_l with $l > m$, having equal residue when divided by 2001: $a_l = a_m \bmod 2001$. So, $a_l - a_m$ is divisible by 2001. But $a_l - a_m = \underbrace{11\ldots1}_{l-m}\underbrace{00\ldots0}_{m} = a_{l-m} \cdot 10^m$. Since 10^m is relatively prime to 2001, a_{l-m} is divisible by 2001.

4. Let $ABCD$ be the rectangle R and $KLMN$ the quadrilateral Q with one vertex on each edge of R, as shown in the upper left of the picture. Starting with R, reflect three times: first reflect in edge BC so that $ABCD$ goes to a rectangle $A'BCD'$; then reflect in edge CD' so that $A'BCD'$ goes into a rectangle $A''B'CD'$; then reflect in edge $A''D'$ so that $A''B'CD'$ goes into a rectangle $A''B''C''D'$. Then, as shown in the picture, $KLMN$ maps successively to quadrilaterals $K'LM'N'$, $K'L'M''N''$, and $K''L''M'''N''$. Now consider the line segment MM'''. Since AM and $A''M'''$ are parallel and congruent, we know that MM''' is parallel and congruent to AA'', so the length $|MM'''|$ of MM''' is twice the length of the diagonal of R. But $|MM'''|$ is at most equal to the sum $|ML| + |LK'| + |K'N''| + |N''M'''|$, which is equal to $|ML| + |LK| + |KN| + |NM|$, the perimeter of Q.



5. Let M be the maximum of the numbers placed on the chessboard, say M is in square S. Observe that all the numbers placed in the neighboring squares of S have to be equal to M as well. Since any square of the chessboard is reachable from S through a chain of neighbours, all the numbers placed on the chessboard are equal to M.

6. First we compute the absolute value of z as follows: $|z|^2 = \left(\frac{-1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1$, so $|z| = 1$. Next compute the square of z as follows: $z^2 = \left(\frac{-1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2 + 2 \cdot \frac{-1}{2} \cdot \frac{\sqrt{3}}{2} i = \frac{-1}{2} + \frac{-\sqrt{3}}{2} i$. Note that $1 + z + z^2 = 0$. Now assume $a + bz + cz^2 = 0$. Subtract $0 = 0c = (1 + z + z^2)c$ to get $(a - c) + (b - c)z = 0$. Therefore $a - c = (c - b)z$ so that $|a - c| = |c - b| \cdot |z| = |c - b|$. Those two sides of the triangle have the same length. Similarly, subtract $0 = 0b = (1 + z + z^2)b$ from $a + bz + cz^2 = 0$ to get $(a - b) + (c - b)z^2 = 0$ so that $|a - b| = |b - c| \cdot |z|^2 = |b - c|$. So, in fact, all three sides have the same length.