Solutions: 2003 Rasor-Bareis Prize Examination

1. Let f_n be the remainder when F_n is divided by 2003; that is, $0 \le f_n \le 2002$ and $F_n - f_n$ is a multiple of 2003. If, for some n, both f_n, f_{n+1} are known, then we may find f_{n+2} and f_{n-1}; namely the remainder mod 2003 of $f_n + f_{n+1}$ and of $f_{n+1} - f_n$, respectively. Now, there are finitely many ordered pairs (x, y) of integers with $0 \le x, y \le 2002$. So if we consider the pairs (f_n, f_{n+1}) as n ranges over the positive integers, there must be repeats. Thus, there are positive integers N, k so that $(f_N, f_{N+1}) = (f_{N+k}, f_{N+k+1})$. As noted above, given a consecutive pair we can go backward, so $(f_{k+1}, f_{k+2}) = (f_1, f_2) = (1, 1)$ and thus $f_k = 0$. That means F_k is divisible by 2003.

2. [Solution I]
Since α, β, γ are the three angles in a triangle, we have $\alpha + \beta + \gamma = \pi$. Then $\cos\gamma = \cos(\pi - (\alpha + \beta)) = -\cos(\alpha + \beta)$, so we have

$$\begin{aligned}
1 &= \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \cos^2\alpha + \cos^2\beta + \cos^2(\alpha + \beta) \\
&= \cos^2\alpha + \cos^2\beta + \big(\cos\alpha\,\cos\beta - \sin\alpha\,\sin\beta\big)^2 \\
&= \cos^2\alpha + \cos^2\beta + \cos^2\alpha\cos^2\beta + \sin^2\alpha\sin^2\beta - 2\cos\alpha\,\cos\beta\,\sin\alpha\,\sin\beta \\
&= \cos^2\alpha + \cos^2\beta + \cos^2\alpha\cos^2\beta + (1 - \cos^2\alpha)(1 - \cos^2\beta) \\
&\qquad\qquad - 2\cos\alpha\,\cos\beta\,\sin\alpha\,\sin\beta \\
&= 1 + 2\cos^2\alpha\cos^2\beta - 2\cos\alpha\,\cos\beta\,\sin\alpha\,\sin\beta \\
&= 1 + 2\cos\alpha\,\cos\beta\big(\cos\alpha\,\cos\beta - \sin\alpha\,\sin\beta\big) \\
&= 1 + 2\cos\alpha\,\cos\beta\,\cos(\alpha + \beta) = 1 - 2\cos\alpha\,\cos\beta\,\cos\gamma.
\end{aligned}$$

So: if $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, then we have $\cos\alpha\cos\beta\cos\gamma = 0$, which means one of the three cosines is 0, which means one of the three angles is $\pi/2$.

[Solution II]
First, $\cos^2\alpha + \cos^2\beta = (1 + \cos 2\alpha)/2 + (1 + \cos 2\beta)/2 = 1 + \cos(\alpha + \beta)\cos(\alpha - \beta)$ and $\cos\gamma = \cos(\pi - (\alpha + \beta)) = -\cos(\alpha + \beta)$, so if $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ then $\cos(\alpha + \beta)\cos(\alpha - \beta) + \cos^2(\alpha + \beta) = 0$, and so $\cos(\alpha + \beta)\big(\cos(\alpha - \beta) + \cos(\alpha + \beta)\big) = -2\cos\gamma\,\cos\alpha\,\cos\beta = 0$. It follows that either $\cos\alpha = 0$, or $\cos\beta = 0$, or $\cos\gamma = 0$ and so, one of the angles is $\pi/2$.

3. We claim the only integer solution is $x = y = z = 0$. Suppose there is an integer solution where at least one is nonzero. Then by the well-ordering property of the positive integers, there is an integer solution x, y, z such that $|x| + |y| + |z| > 0$ is as small as possible. Then $x^3 = 2y^3 + 4z^3$ is even, so x is even; say $x = 2X$ for some integer X. Divide by 2 to get $4X^3 - y^3 - 2z^3 = 0$. Therefore $y^3 = 4X^3 - 2z^3$ is even, so y is even, say $y = 2Y$. Divide by 2 again to get $2X^3 - 4Y^3 - z^3 = 0$. Then $z^3 = 2X^3 - 4Y^3$ is even, so z is even, say $z = 2Z$. Divide by 2 a third time to get $X^3 - 2Y^3 - 4Z^3 = 0$. But that means X, Y, Z is an integer solution of the original equation with $|X| + |Y| + |Z|$ smaller than $|x| + |y| + |z|$. This contradiction shows that there is no nonzero solution.

4. If x, y are real, then the complex number $a = x + iy$ has complex conjugate $\overline{a} = x - iy$. Then we have $a\,\overline{a} = |a|^2$. So if $|a| = 1$, then $\overline{a} = 1/a$. Now suppose a, b, c are complex numbers and $|a| = |b| = |c| = 1$. Then

$$\begin{aligned}
|ab + ac + bc| = 1 \cdot |ab + ac + bc| &= \frac{1}{|abc|} \cdot |ab + ac + bc| = \left|\frac{ab}{abc} + \frac{ac}{abc} + \frac{bc}{abc}\right| \\
&= \left|\frac{1}{c} + \frac{1}{b} + \frac{1}{a}\right| = \left|\,\overline{a} + \overline{b} + \overline{c}\,\right| = \left|\,\overline{a + b + c}\,\right| = |a + b + c|.
\end{aligned}$$

5. Let n be a positive integer; since sine is a periodic function with period 2π, when computing $\sin(2\pi en!)$ we are only interested in the fractional part a_n of $en!$. Now

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{(n-1)!} + \frac{1}{n!} + \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \ldots,$$

and thus

$$\begin{aligned} en! &= n! + \frac{n!}{1!} + \frac{n!}{2!} + \ldots + \frac{n!}{(n-1)!} + 1 + \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \ldots \\ &= M + \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \ldots, \end{aligned}$$

where M is integer. Hence, $\sin(2\pi en!) = \sin(2\pi a_n)$, where

$$a_n = \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \cdots.$$

We will estimate a_n by comparison with a geometric series:

$$\frac{1}{n+1} < a_n < \frac{1}{n+1} + \frac{1}{(n+1)^2} + \ldots = \frac{\frac{1}{n+1}}{1 - \frac{1}{n+1}} = \frac{1}{n}.$$

It follows that $a_n \to 0$, and so, $\lim_{n\to\infty} \frac{\sin(2\pi a_n)}{2\pi a_n} = \lim_{x\to 0} \frac{\sin x}{x} = 1$. Hence,

$$\lim_{n\to\infty} n\sin(2\pi a_n) = \lim_{n\to\infty} (2\pi n a_n) \cdot \lim_{n\to\infty} \frac{\sin(2\pi a_n)}{2\pi a_n} = 2\pi \lim_{n\to\infty} n a_n.$$

Since $\frac{1}{n+1} < a_n < \frac{1}{n}$, by the squeeze theorem $\lim_{n\to\infty} n a_n = 1$, and

$$\lim_{n\to\infty} n\sin(2\pi en!) = \lim_{n\to\infty} n\sin(2\pi a_n) = 2\pi.$$

6. [See the Gordon solutions.]