Rasor-Bareis Solutions

1. Find the value of $\lim_{n\to+\infty} \dfrac{1+2^2+\cdots+n^n}{n^n}$ if it exists. If it does not exist, say why.

The limit is 1. First,

$$\frac{1+2^2+\cdots+n^n}{n^n} > \frac{n^n}{n^n} = 1.$$

On the other hand,

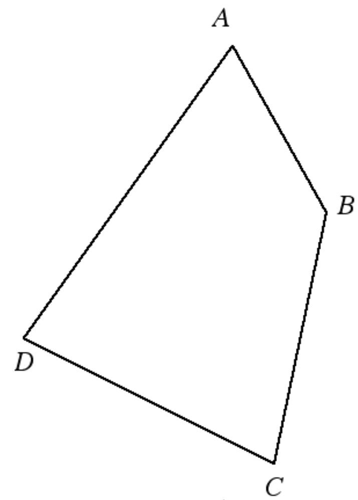
$$\frac{1+2^2+\cdots+n^n}{n^n} \le (n-2)\frac{n^{n-2}}{n^n} + \frac{n^{n-1}}{n^n} + \frac{n^n}{n^n} = \frac{n-2}{n^2} + \frac{1}{n} + 1,$$

and this converges to 1. So our sequence is "squeezed" between two sequences with limit 1, so it has limit 1.

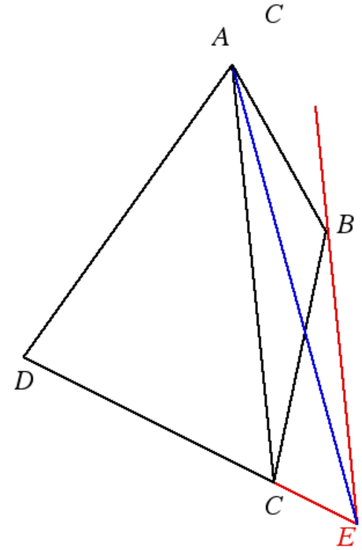
NOTE: Many contestants tried to do this limit of a sum as a sum of limits. It is not enough to show all terms k^k/n^n but the last go to 0, because the number of terms goes to ∞.

2. Let Q be a convex quadrilateral in the plane. Show that a line can be constructed, using straight-edge and compass only, that divides Q into two regions of equal area.

Label the vertices A, B, C, D in order. Draw the diagonal line AC. Since Q is convex, vertices B and D are on opposite sides of line AC.

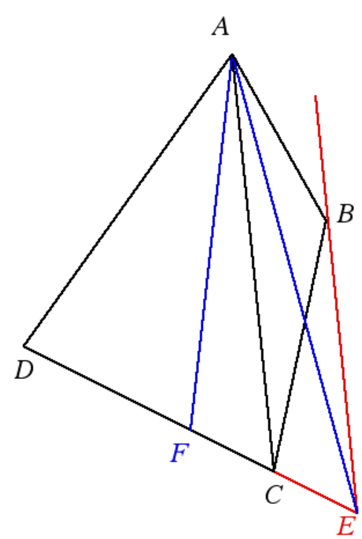


Draw the line through B parallel to AC and draw the line extending DC. Let E be the point where these two lines intersect. These lines do intersect as they are not parallel, since DC and AC are not parallel.



Note that triangles ABC and AEC have the same area, since they have the same base AC and congruent altitudes from that base.

Construct the midpoint F of segment DE. Then triangles AFD and AFE have the same area, since they have congruent bases DF and FE and the same altitude.



Now there are two cases, depending on whether F lies between D and C or F lies between C and E. If F lies between D and C (or even if F coincides with C), then the line AF is the required line. Triangle AFD has the same area as triangle AFE, which has area equal to the sum of the areas AFC and ACE, which in turn is equal to the sum of the areas of AFC and ABC, or the area of quadrilateral $ABCF$.

The other case, when F lies between C and E, means that triangle ACD is less than half of the total area of $ABCD$. In this case, repeat the construction starting with the line through D parallel to AC, then proceed as before with B and D interchanged.

3. Let f be a real-valued function such that $f(2003) = 2\pi$ and $|f(x) - f(y)|^2 \leq |x - y|^3$ for all real numbers x and y. Compute $f(2004)$.

From the given inequality, we get

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq |x - y|^{1/2}, \qquad x \neq y.$$

So if we fix x and let $y \to x$, we conclude that the derivative $f'(x)$ exists and is equal to 0. Since the derivative is zero everywhere, the function is constant. So $f(2004) = f(2003) = 2\pi$.

4. Let $P(x)$ be a nonconstant polynomial with integer coefficients. Is it possible that $P(n)$ is a prime number for all integers n?

Solution I. No, it is not possible. Assume P is a polynomial with integer coefficients such that $P(x)$ is prime for all integers x. The "constant term" of $P(x)$ is $P(0)$, so it is prime, call it p. If $x = kp$ is an integer multiple of p, then all the terms in $P(kp)$ except the constant term are divisible by p, so $P(kp)$ is divisible by p, and therefore (since it is supposed to be prime) equal to p. So, since $P(x)$ has the same value p for infinitely many x, we know that P is constant. [If you interpret the problem to allow $-p$ also to be called "prime", then note that all values $P(kp)$ are divisible by p and prime, so all those values are either p or $-p$, so at least one of these values is achieved infinitely many times, and again we conclude that P must be constant.]

Solution II. Let a natural number n_0 be such that for any $m > n_0$ one has: $P(m+1) > P(m)$. (This is possible since our polynomial is non-constant and takes positive values, so it is increasing from some point on). Take any $m > n_0$ and let p be a prime such that $P(m) = p$. We can see that $P(m+p)$ is divisible by p by expanding the powers of $m+p$. This, together with the fact that $P(m+p) > P(m) = p$, shows that $P(m + p)$ is not a prime number. Contradiction.

5. Given any selection of 1004 distinct integers from the set $\{1, 2, \ldots, 2004\}$, show that some three of the selected integers have the property that one is the sum of the other two.

Solution I. Let m be the largest of the selected integers. This leaves 1003 selected integers in $\{1, \ldots, m-1\}$. Consider the pairs of distinct integers in $\{1, \ldots, m-1\}$ that add to m: these pairs are $(1, m-1)$, $(2, m-2)$, $(3, m-3)$, etc. If m is even, then there are $(m-2)/2$ pairs, and one number $m/2$ left over. If m is odd, there are $(m-1)/2$ pairs with nothing left over. There are 1003 selected integers in $\{1, \ldots, m-1\}$, and $1003 = (2006-2)/2+1 > (m-2)/2+1$ if m is even and $1003 = (2007-1)/2 > (m-1)/2$ if m is odd, so at least one of the pairs has both components selected. This pair, together with m, gives us a selected triple such that one of the integers is the sum of the other two.

Solution II. Let A be the set of selected integers and let m be the largest element in A. Let $B = \{m - a \mid a \in A, a \neq m\}$. Then $|A| = 1004$ and $|B| = 1003$. Hence $A \cap B$ contains at least 3 elements, say a, b, and c. So, for some x, y, and $z \in A$, we have

$$(*) \qquad a = m - x, \; b = m - y, \; \text{ and } \; c = m - z.$$

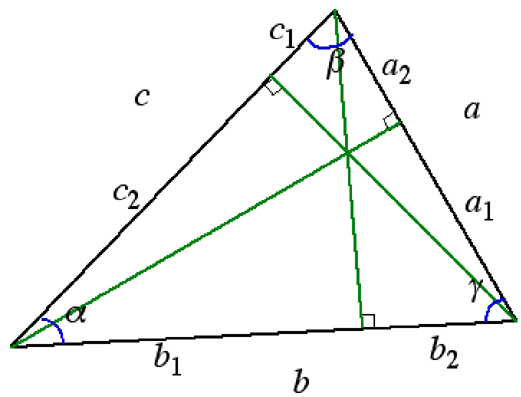
None of a, b, or c can equal m. Also, since the a, b, and c are distinct, only one of them can equal $m/2$. Hence at least two(!) of the equations in $(*)$ involve three distinct elements of A (these two equations can be the same equation written in different order).

6. Let α, β, γ be the angles of a triangle. Show that $\cos\alpha \cdot \cos\beta \cdot \cos\gamma \le \frac{1}{8}$.

Solution I. (*Submitted by Donald Seelig*) First, if the triangle is obtuse then $\cos\alpha \cdot \cos\beta \cdot \cos\gamma < 0$ and if the triangle is right, then $\cos\alpha \cdot \cos\beta \cdot \cos\gamma = 0$. So assume the triangle is acute.

The altitude from the vertex with angle α divides the opposite side a into two parts a_1, a_2. The altitude from the vertex with angle β divides the opposite side b into two parts b_1, b_2. The altitude from the vertex with angle γ divides the opposite side c into two parts c_1, c_2. From right triangle trigonometry, we get

$$\cos\alpha = \frac{b_1}{c} = \frac{c_2}{b}, \qquad \cos\beta = \frac{a_2}{c} = \frac{c_1}{a}, \qquad \cos\gamma = \frac{b_2}{a} = \frac{a_1}{b}.$$



Then $a_1 b_1 c_1 = abc\cos\alpha\cos\beta\cos\gamma = a_2 b_2 c_2$, and algebraic manipulation gives us:

$$\frac{1}{\cos\alpha\cos\beta\cos\gamma} = 2 + \frac{a_2}{a_1} + \frac{a_1}{a_2} + \frac{b_2}{b_1} + \frac{b_1}{b_2} + \frac{c_2}{c_1} + \frac{c_1}{c_2}.$$

Now note that for $x > 0$ we have $x + 1/x \ge 2$ (this follows from $(x-1)^2 \ge 0$), so

$$\frac{1}{\cos\alpha\cos\beta\cos\gamma} \ge 2 + 2 + 2 + 2 = 8.$$

Solution II. Note that

$$\cos\gamma = \cos(\pi - (\alpha+\beta)) = -\cos(\alpha+\beta),$$
$$2\cos\alpha\cos\beta = \cos(\alpha-\beta) + \cos(\alpha+\beta).$$

Hence:

$$\begin{aligned}
&8\cos\alpha\cos\beta\cos\gamma - 1 \\
&\qquad = -4\cos(\alpha+\beta)\big[\cos(\alpha+\beta) + \cos(\alpha-\beta)\big] - \big[\cos^2(\alpha-\beta) + \sin^2(\alpha-\beta)\big] \\
&\qquad = -\big[2\cos(\alpha+\beta) + \cos(\alpha-\beta)\big]^2 - \sin^2(\alpha-\beta) \le 0.
\end{aligned}$$

Solution III. For any real θ, the maximum of the function

$$f(x) = \cos(x)\cos(\theta - x) = \frac{1}{2}\big[\cos(\theta) + \cos(2x - \theta)\big]$$

is reached when $2x - \theta = 0$, i.e. when $x = \theta - x$. Thus, when α, β, γ are angles of a triangle, for any fixed γ the maximum of $\cos(\alpha)\cos(\beta)\cos(\gamma) = \cos(\alpha)\cos(\pi - \gamma - \alpha)\cos(\gamma)$ is reached when $\alpha = \pi - \gamma - \alpha = \beta$. Hence, the maximum of $F(\alpha, \beta, \gamma) = \cos(\alpha)\cos(\beta)\cos(\gamma)$ is reached when $\alpha = \beta = \gamma = \pi/3$. (If, say, $\beta \ne \gamma$, then $F(\alpha, (\beta+\gamma)/2, (\beta+\gamma)/2) > F(\alpha, \beta, \gamma)$.)