Solutions: 2005 Rasor-Bareis Prize Examination

1. Let A, B, C, D, E, F denote the lengths of the six edges of the tetrahedron. Up to relabeling, the perimeters of the four faces are

$$A+B+C, \quad A+E+F, \quad B+F+D, \quad C+D+E,$$

and we have

$$A+B+C = A+E+F = B+F+D = C+D+E.$$

Then

$$2A+B+C+E+F = A+B+C+A+E+F = B+F+D+C+D+E = 2D+B+C+E+F,$$

so $A = D$. Similarly,

$$2B+A+C+D+F = A+B+C+B+F+D = A+E+F+C+D+E = 2E+A+C+D+F,$$

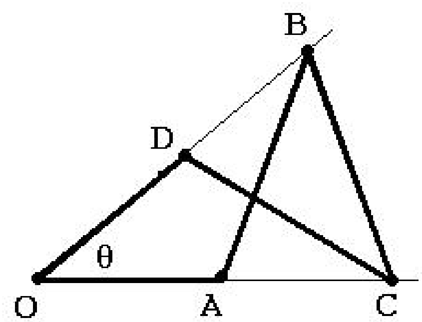
so $B = E$. Finally,

$$2C+A+B+D+E = A+B+C+C+D+E = A+E+F+B+F+D = 2F+A+B+D+E,$$

so $C = F$. Hence, each face of the tetrahedron has one side of length A, one side of length B, and one side of length C. Therefore, since triangles with identical edge lengths are congruent, the faces of the tetrahedron are all congruent.

2. Claim: $\theta = \pi/5$ $(= 36°)$.

Proof. Let A, B, C, and D be the points other than O traversed in order by the five steps. Then $|OA| = |AB| = |BC| = |CD| = |DO|$. Since $|AB| = |CD|$, $|OD| = |OA|$ and the triangles OAB and ODC have the common angle $\angle AOB$, they are congruent; thus $|OC| = |OB|$ and so $\triangle OBC$ is isosceles. We now compute angles inside the triangle OBC. $\angle BOC = \theta$, as given. Since $\triangle OAB$ is isosceles, $\angle OBA = \theta$, so $\angle OAB = \pi - 2\theta$ and $\angle BAC = 2\theta$. Since $\triangle ABC$ is isosceles, $\angle BCA = 2\theta$. Since $\triangle OBC$ is isosceles, $\angle OBC = 2\theta$. Thus the angles of $\triangle OBC$ add up to 5θ, and so, $\theta = \pi/5$.



3. Let us change the variable x to $t = x - 2005/2$, and define $g(t) = f(x) = f(t + 2005/2)$. Then

$$\begin{aligned} g(t) &= \left|\left(t+\tfrac{2005}{2}\right)\left(t+\tfrac{2003}{2}\right)\cdots\left(t+\tfrac{1}{2}\right)\left(t-\tfrac{1}{2}\right)\cdots\left(t-\tfrac{2003}{2}\right)\left(t-\tfrac{2005}{2}\right)\right| \\ &= \left|\left(t^2-\left(\tfrac{2005}{2}\right)^2\right)\left(t^2-\left(\tfrac{2003}{2}\right)^2\right)\cdots\left(t^2-\left(\tfrac{1}{2}\right)^2\right)\right|. \end{aligned}$$

For $x \in [1002, 1003]$ we have $t \in \left[\frac{-1}{2}, \frac{1}{2}\right]$, and on this interval

$$g(t) = \left(\left(\tfrac{2005}{2}\right)^2 - t^2\right)\left(\left(\tfrac{2003}{2}\right)^2 - t^2\right)\dots\left(\left(\tfrac{1}{2}\right)^2 - t^2\right).$$

Each factor in this product attains maximal value at $t = 0$. Thus $g(t)$ attains maximal value M at $t = 0$, and $M = g(0) = \left(\frac{2005}{2}\right)^2\left(\frac{2003}{2}\right)^2 \dots \left(\frac{1}{2}\right)^2 = (1 \cdot 3 \cdot \ldots \cdot 2005)^2/2^{2006}$.

4. Solution 1. Let $2b^3 + 5 = x^2$, for some integers $b \geq 6$ and x. Then x^2 is odd because $2b^2 + 5$ is odd, and so, x is odd. So $x^2 = 4y^2 + 4y + 1$, for some integer y. Subtracting 5 from both sides of $2b^3 + 5 = 4y^2 + 4y + 1$, we get

$$2b^3 = 4y^2 + 4y - 4 = 4(y^2 + y - 1),$$

that is, $b^3 = 2(y^2 + y - 1)$. Hence b is even, so the left side is divisible by 4. However, $y^2 + y - 1$ is odd; contradiction.

Solution 2. Assume that $2b^3 + 5 = x^2$, for some integers b and x. The integer x is odd because the left hand side of the equation is odd. Rewrite the equation as $2b^3 + 4 = x^2 - 1 = (x-1)(x+1)$. The numbers $x - 1$ and $x + 1$ are successive even integers, thus $(x-1)(x+1)$ is divisible by 8. On the other hand, if b is odd, then $2b^3$ is not divisible by 4 and thus $2b^3 + 4$ is not divisible by 4; if b is even, then $2b^3$ is divisible by 8 and thus $2b^3 + 4$ is not divisible by 8. Contradiction.

5. The function $\sqrt{x}$ increases to infinity as $x \to \infty$, but the derivative $1/(2\sqrt{x})$ of this function decreases to 0. As a result, the distances between successive points of the sequence $\sqrt{n}$ tend to zero. Indeed, by the mean value theorem, for any $n \in \mathbb{N}$, $\sqrt{n+1} - \sqrt{n} = 1/(2\sqrt{z})$ for some $z \in (n, n+1)$, which tends to 0 as $n \to \infty$. It follows that the sequence $\langle\sqrt{n}\rangle$ of fractional parts of $\sqrt{n}$ is dense in $[0, 1]$.

Here is a more rigorous proof: Let $\varepsilon > 0$. Choose a positive integer N such that $1/(2N) < \varepsilon$. Put $a_k = \left\langle\sqrt{N^2 + k}\right\rangle = \sqrt{N^2 + k} - N$ for $k = 0, \dots, 2N$ and $a_{2N+1} = \sqrt{N^2 + 2N + 1} - N = 1$. Now $0 = a_0 < \ldots < a_{2N+1} = 1$, so there exists a $k \in 0, \dots, 2N$ such that $\frac{1}{2005} \in [a_k, a_{k+1}]$. Hence it suffices to show that $a_{k+1} - a_k < \varepsilon$ for all $k = 0, \dots, 2N$.

For any $k \in \{0, \dots, 2N\}$,

$$a_{k+1} - a_k = \left(\sqrt{N^2 + k + 1} - N\right) - \left(\sqrt{N^2 + k} - N\right) = \sqrt{N^2 + k + 1} - \sqrt{N^2 + k}.$$

By the mean value theorem,

$$\sqrt{N^2 + k + 1} - \sqrt{N^2 + k} = 1/(2\sqrt{z}),$$

for some $z \in \left[N^2 + k, N^2 + k + 1\right]$. Since $z > N^2 + k$, we have $\sqrt{z} > N$, and $1/(2\sqrt{z}) < 1/(2N) < \varepsilon$.

6. For each $n \geq 1$, there are at most $n^{1/2}$ squares and at most $n^{1/3}$ cubes contained in $\{1, \dots, n\}$, so there are at most $n^{1/2} \cdot n^{1/3} = n^{5/6}$ numbers of the form $x^2 + y^3$ contained in $\{1, \dots, n\}$, where x and y are nonnegative integers. Since $n - n^{5/6} \longrightarrow \infty$ as $n \longrightarrow \infty$, the result follows.