1. Let $A, B, C, D, E, F$ denote the lengths of the six edges of the tetrahedron. Up to relabeling, the perimeters of the four faces are

$$A + B + C, \quad A + E + F, \quad B + F + D, \quad C + D + E,$$

and we have

$$A + B + C = A + E + F = B + F + D = C + D + E.$$

Then

$$2A + B + C + E + F = A + B + C + A + E + F = B + F + D + C + D + E = 2D + B + C + E + F,$$

so $A = D$. Similarly,

$$2B + A + C + D + F = A + B + C + B + F + D = A + E + F + C + D + E = 2E + A + C + D + F,$$

so $B = E$. Finally,

$$2C + A + B + D + E = A + B + C + C + D + E = A + E + F + B + F + D = 2F + A + B + D + E,$$

so $C = F$. Hence, each face of the tetrahedron has one side of length $A$, one side of length $B$, and one side of length $C$. Therefore, since triangles with identical edge lengths are congruent, the faces of the tetrahedron are all congruent.

2. Claim: $\theta = \pi/5$ ($= 36^\circ$).

Proof. Let $A$, $B$, $C$, and $D$ be the points other than $O$ traversed in order by the five steps. Then $|OA| = |AB| = |BC| = |CD| = |DO|$. Since $|AB| = |CD|$, $|OD| = |OA|$ and the triangles $OAB$ and $ODC$ have the common angle $\angle AOB$, they are congruent; thus $|OC| = |OB|$ and so $\triangle OBC$ is isosceles. We now compute angles inside the triangle $OBC$. $\angle BOC = \theta$, as given. Since $\triangle OAB$ is isosceles, $\angle OBA = \theta$, so $\angle OAB = \pi - 2\theta$ and $\angle BAC = 2\theta$. Since $\triangle ABC$ is isosceles, $\angle BCA = 2\theta$. Since $\triangle OBC$ is isosceles, $\angle OBC = 2\theta$. Thus the angles of $\triangle OBC$ add up to $5\theta$, and so, $\theta = \pi/5$.

3. Let us change the variable $x$ to $t = x - 2005/2$, and define $g(t) = f(x) = f(t + 2005/2)$. Then

$$g(t) = \left| \left(t + \frac{2005}{2}\right)\left(t + \frac{2003}{2}\right)\ldots\left(t + \frac{1}{2}\right)\left(t - \frac{1}{2}\right)\ldots\left(t - \frac{2003}{2}\right)\left(t - \frac{2005}{2}\right) \right|$$

$$= \left| \left(t^2 - \left(\frac{2005}{2}\right)^2\right)\left(t^2 - \left(\frac{2003}{2}\right)^2\right)\ldots\left(t^2 - \left(\frac{1}{2}\right)^2\right) \right|.$$
For $x \in [1002, 1003]$ we have $t \in \left[\frac{-1}{2}, \frac{1}{2}\right]$, and on this interval

$$g(t) = \left(\left(\frac{2005}{2}\right)^2 - t^2\right)\left(\left(\frac{2003}{2}\right)^2 - t^2\right) \ldots \left(\frac{1}{2} - t^2\right).$$

Each factor in this product attains maximal value at $t = 0$. Thus $g(t)$ attains maximal value $M$ at $t = 0$, and $M = g(0) = \left(\frac{2005}{2}\right)^2\left(\frac{2003}{2}\right)^2 \ldots \left(\frac{1}{2}\right)^2 = (1 \cdot 3 \ldots \cdot 2005)^2/2^{2006}$.

4. Solution 1. Let $2b^3 + 5 = x^2$, for some integers $b \geq 6$ and $x$. Then $x^2$ is odd because $2b^2 + 5$ is odd, and so, $x$ is odd. So $x^2 = 4y^2 + 4y + 1$, for some integer $y$. Subtracting 5 from both sides of $2b^3 + 5 = 4y^2 + 4y + 1$, we get

$$2b^3 = 4y^2 + 4y - 4 = 4(y^2 + y - 1),$$

that is, $b^3 = 2(y^2 + y - 1)$. Hence $b$ is even, so the left side is divisible by 4. However, $y^2 + y - 1$ is odd; contradiction.

Solution 2. Assume that $2b^3 + 5 = x^2$, for some integers $b$ and $x$. The integer $x$ is odd because the left hand side of the equation is odd. Rewrite the equation as $2b^3 + 4 = x^2 - 1 = (x - 1)(x + 1)$. The numbers $x - 1$ and $x + 1$ are successive even integers, thus $(x - 1)(x + 1)$ is divisible by 8. On the other hand, if $b$ is odd, then $2b^3$ is not divisible by 4 and thus $2b^3 + 4$ is not divisible by 4; if $b$ is even, then $2b^3$ is divisible by 8 and thus $2b^3 + 4$ is not divisible by 8. Contradiction.

5. The function $\sqrt{x}$ increases to infinity as $x \to \infty$, but the derivative $1/(2\sqrt{x})$ of this function decreases to 0. As a result, the distances between successive points of the sequence $\sqrt{n}$ tend to zero. Indeed, by the mean value theorem, for any $n \in \mathbb{N}$, $\sqrt{n+1} - \sqrt{n} = 1/(2\sqrt{z})$ for some $z \in (n, n+1)$, which tends to 0 as $n \to \infty$. It follows that the sequence $\langle \sqrt{n} \rangle$ of fractional parts of $\sqrt{n}$ is dense in $[0, 1]$.

Here is a more rigorous proof: Let $\varepsilon > 0$. Choose a positive integer $N$ such that $1/(2N) < \varepsilon$. Put $a_k = \langle \sqrt{N^2 + k} \rangle = \sqrt{N^2 + k} - N$ for $k = 0, \ldots, 2N$ and $a_{2N+1} = \sqrt{N^2 + 2N + 1} - N = 1$. Now $0 = a_0 < \ldots < a_{2N+1} = 1$, so there exists a $k \in 0, \ldots, 2N$ such that $\frac{1}{2005} \in [a_k, a_{k+1}]$. Hence it suffices to show that $a_{k+1} - a_k < \varepsilon$ for all $k = 0, \ldots, 2N$.

For any $k \in \{0, \ldots, 2N\}$,

$$a_{k+1} - a_k = (\sqrt{N^2 + k + 1} - N) - (\sqrt{N^2 + k} - N) = \sqrt{N^2 + k + 1} - \sqrt{N^2 + k}.$$

By the mean value theorem,

$$\sqrt{N^2 + k + 1} - \sqrt{N^2 + k} = 1/(2\sqrt{z}),$$

for some $z \in [N^2 + k, N^2 + k + 1]$. Since $z > N^2 + k$, we have $\sqrt{z} > N$, and $1/(2\sqrt{z}) < 1/(2N) < \varepsilon$.

6. For each $n \geq 1$, there are at most $n^{1/2}$ squares and at most $n^{1/3}$ cubes contained in $\{1, \ldots, n\}$, so there are at most $n^{1/2} \cdot n^{1/3} = n^{5/6}$ numbers of the form $x^2 + y^3$ contained in $\{1, \ldots, n\}$, where $x$ and $y$ are nonnegative integers. Since $n - n^{5/6} \to \infty$ as $n \to \infty$, the result follows.