Rasor-Bareis solutions

1. Define $a_0 = 1$ and $a_{n+1} = a_n/(1 + na_n)$. Determine a_{2007}.

Solution I.
Define $b_n = \frac{1}{a_n}$, $n = 0, 1, 2, \ldots$ Then the numbers b_n satisfy the recursive formula

$$b_{n+1} = \frac{1}{a_{n+1}} = \frac{1 + na_n}{a_n} = \frac{1}{a_n} + n = b_n + n.$$

Hence, for any n, $b_n = 1+0+1+2+\ldots+n-1 = 1+\frac{n(n-1)}{2}$, and $a_n = \frac{1}{1+n(n-1)/2}$. In particular, $a_{2007} = \frac{1}{2013022}$.

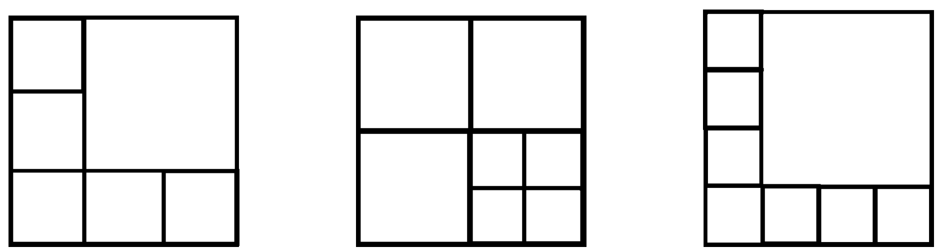
Solution II.
We will prove by induction that $a_n = \frac{1}{1+n(n-1)/2}$ for all n. This is so for $n = 0$, and if this holds for some n then

$$a_{n+1} = \frac{a_n}{1 + na_n} = \frac{\frac{1}{1+n(n-1)/2}}{1 + n\frac{1}{1+n(n-1)/2}} = \frac{1}{1 + \frac{n(n-1)}{2} + n} = \frac{1}{1 + \frac{(n+1)n}{2}},$$

thus it holds for $n + 1$.

2. Show that for any integer $n \geq 6$, a square in the plane can be dissected into n squares.

Observe first that if a square S is dissected into n squares $S_1, S_2, \ldots, S_n$, then replacing S_n by four congruent squares $S_{n1}, S_{n2}, S_{n3}, S_{n4}$ creates a partition into $n + 3$ squares. So it is enough to show that S can be dissected into 6, 7, and 8 squares. This can be done, for example, as follows:



Note that the picture on the *RB* sheet gives another dissection into 8 squares.

3. Find

$$\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n+1}+\cdots+\frac{1}{2n}\right).$$

Solution I.
Represent

$$\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}=\frac{1}{n}+\frac{1}{n}\left(\frac{1}{1+\frac{1}{n}}+\frac{1}{1+\frac{2}{n}}+\ldots+\frac{1}{1+\frac{n}{n}}\right).$$

Note that

$$L_n=\frac{1}{n}\left(\frac{1}{1+\frac{1}{n}}+\frac{1}{1+\frac{2}{n}}+\ldots+\frac{1}{1+\frac{n}{n}}\right)$$

is (the lower) Riemann sum of the function $\frac{1}{1+x}$ on the interval $[0,1]$ corresponding to the partition $\left\{0,\frac{1}{n},\frac{2}{n},\ldots,1\right\}$ of this interval. Hence, $\lim_{n\to\infty}L_n=\int_0^1\frac{dx}{1+x}=\log 2$. Since $\lim_{n\to\infty}\frac{1}{n}=0$, we obtain $\lim_{n\to\infty}\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}=\log 2$.

Solution II.
Let $H_n=1+\frac{1}{2}+\ldots+\frac{1}{n}$, $n=1,2,\ldots$ It is well known(?) (and/or can be easily proved!) that the sequence $H_n-\log n$ has a finite limit γ. (γ is called *Euler's constant*, and is approximately equal 0.577.) We therefore have

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}\\
&\quad=\lim_{n\to\infty}(H_{2n}-H_{n-1})\\
&\quad=\lim_{n\to\infty}\big((H_{2n}-\log 2n)-(H_{n-1}-\log(n-1))+(\log 2n-\log(n-1))\big)\\
&\quad=\lim_{n\to\infty}(H_{2n}-\log 2n)-\lim_{n\to\infty}(H_{n-1}-\log(n-1))+\lim_{n\to\infty}\big(\log 2n-\log(n-1)\big)\\
&\quad=\gamma-\gamma+\lim_{n\to\infty}\log\tfrac{2n}{n-1}=\log 2.
\end{aligned}$$

4. Show that given any 1004 elements from $\{2, 3, \ldots, 2007\}$, some two are relatively prime.

There is an integer n such that both n and $n+1$ are chosen. (Indeed, if each chosen integer were followed by a non-chosen one, then the total number of elements in the set $\{2, 3, \ldots, 2007\}$ would be $\geq 1004+1003 > 2006$.) This solves the problem since n and $n+1$ cannot have a common divisor different from ± 1.

5. Determine the largest constant $k > 0$ such that for all complex numbers z_1, z_2, z_3 with $|z_1| = |z_2| = |z_3| = 1$, one has

$$|z_1 z_2 + z_2 z_3 + z_3 z_1| \geq k\,|z_1 + z_2 + z_3|.$$

Observe that the condition $|z_1| = |z_2| = |z_3| = 1$ implies that $|z_1 z_2 z_3| = 1$ and that $z_i^{-1} = \bar{z}_i$, $i = 1, 2, 3$. We claim that $\left|z_1 z_2 + z_2 z_3 + z_3 z_1\right| = |z_1 + z_2 + z_3|$. Indeed,

$$\left|z_1 z_2 + z_2 z_3 + z_3 z_1\right| = \left|\frac{z_1 z_2 z_3}{z_3} + \frac{z_1 z_2 z_3}{z_2} + \frac{z_1 z_2 z_3}{z_1}\right| = |z_1 z_2 z_3|\left|z_3^{-1} + z_2^{-1} + z_1^{-1}\right|$$

$$= \left|\bar{z}_3 + \bar{z}_2 + \bar{z}_1\right| = \left|\,\overline{z_3 + z_2 + z_1}\,\right| = |z_1 + z_2 + z_3|.$$

It follows that the largest k satisfying the inequality $\left|z_1 z_2 + z_2 z_3 + z_3 z_1\right| \geq k|z_1 + z_2 + z_3|$ for all z_1, z_2, z_3 of modulus 1 is $k = 1$.

6. Prove that if a parallelogram is inscribed into a circle (all four vertices on the circle), then it must be a rectangle.

Solution I

Call the parallelogram in question $ABCD$ and the circle K. By some combination of rotating, translating, reflecting, and scaling K, we may assume without loss of generality that K is centered at the origin and has radius one, and that side AB is vertical with A lying above B. Denote by a the common x-coordinate of A and B. Then the length of AB is $2\sqrt{1-a^2}$. Since $ABCD$ is a parallelogram, CD is parallel to AB. (and D lies above C) Denote by b the common x-coordinate of C and D. Then the length of CD is $2\sqrt{1-b^2}$. Since $ABCD$ is a parallelogram, the lengths of AB and CD must be equal, implying that $a^2 = b^2$ or that $a = \pm b$. If $a = b$, then $A = D$ and $B = C$, a contradiction. Therefore, $b = -a$, the common y-coordinate of A and D is $\sqrt{1-a^2}$, and the common y-coordinate of B and C is $-\sqrt{1-a^2}$, implying that $ABCD$ is a rectangle.

Solution II

Since AB and CD are chords of K of equal length, the associated arcs $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$ have equal length as well. This implies that the arcs subtended by angles $\angle ABC$ and $\angle BCD$ have equal length, which implies that $\angle ABC \cong \angle BCD$. Since $\angle ABC$ and $\angle BCD$ are adjacent angles of a parallelogram, they sum to $180\,\text{deg}$, and so are both right angles. This implies that $ABCD$ is a rectangle.

Solution III

The angles $\angle ABC$ and $\angle ADC$, being on opposite sides of the chord AC, are supplementary. They are also equal, being opposite angles of a parallelogram. Therefore, both are right angles. Similarly, angles A and C are right angles.