Rasor-Bareis solutions

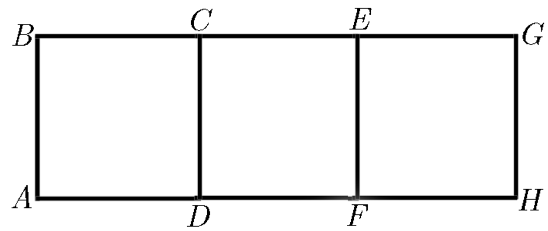
1. Solve: $\sin^{2008} x + \cos^{2008} x = 1$.

We know $\sin^2 x + \cos^2 x = 1$. But if $0 < |u| < 1$, then $u^{2008} < u^2$. So if either $\sin^2 x$ or $\cos^2 x$ is not equal to 0 or 1, we get

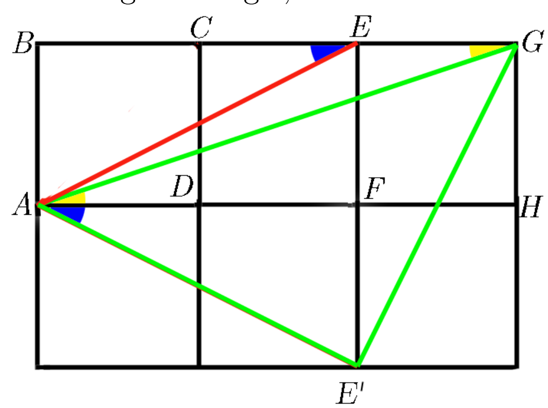
$$\sin^{2008} x + \cos^{2008} x < \sin^2 x + \cos^{2008} x < \sin^2 x + \cos^2 x = 1,$$

so $\sin^{2008} x + \cos^{2008} x < 1$. Therefore: if $\sin^{2008} x + \cos^{2008} x = 1$, then one of $|\sin x|$ or $|\cos x|$ is 1, and so the other is 0. These solutions are $x = n\pi/2$ where n is an integer.

2. Let three adjacent squares be given, as in the diagram.
Show that $\angle ACB + \angle AEB + \angle AGB = 90°$.



Add another row of squares as shown. Then $\angle AEB = \angle E'AF$ and $\angle AGB = \angle GAH$. Since AE' and GE' have the same length and are orthogonal, it follows that $\triangle GAE'$ is an isosceles right triangle, so $\angle GAE'$ is 45 degrees.



Triangles ACE and GCA are similar because the side lengths of ACE are $\sqrt{2}$, 1, and $\sqrt{5}$, while those of GCA are 2, $\sqrt{2}$, and $\sqrt{10}$. Hence $\angle AGB = \angle CAE$. Thus, $\angle AEB + \angle AGB = \angle EAF + \angle CAE = 45°$.

3. Note that 2 can be written as a sum of the reciprocals of four distinct positive integers:

$$2 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{6}.$$

Can 2 be written as a sum of the reciprocals of 2008 distinct positive integers:

$$2 = \frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_{2008}} \qquad ?$$

I claim, in fact, 2 can be written as a sum of the reciprocals of k distinct positive integers for any $k \geq 4$. Indeed,

$$2 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{6}, \qquad 2 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{9} + \frac{1}{18},$$

and the last term $1/(2N)$ can always be replaced by $1/(3N) + 1/(6N)$.

4. Find all rational functions $f(x)$ such that $f(x^2 - x) = f(x^2 + x)$ for all real x.

The constant functions f are exactly the rational functions satisfying this condition. Clearly constant functions do satisfy $f(x^2 - x) = f(x^2 + x)$.

Conversely, assume f is a rational function that satisfies $f(x^2 - x) = f(x^2 + x)$. Note $x^2 - x = (x-1)x$ and $x^2 + x = x(x+1)$. So $f(0 \cdot 1) = f(1 \cdot 2) = f(2 \cdot 3) = \cdots$. The rational function f takes the value $f(0)$ infinitely many times, so f is constant.

5. Let $x_1, x_2, \cdots, x_n$ be distinct integers > 1. Prove:

$$\left(1-\frac{1}{x_1^2}\right)\left(1-\frac{1}{x_2^2}\right)\cdots\left(1-\frac{1}{x_n^2}\right) > \frac{1}{2}.$$

$$\left(1-\frac{1}{x_1^2}\right)\left(1-\frac{1}{x_1^2}\right)\cdots\left(1-\frac{1}{x_1^2}\right) \geq \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{(n+1)^2}\right)$$
$$= \left(\frac{2^2-1}{2^2}\right)\left(\frac{3^2-1}{3^2}\right)\cdots\left(\frac{(n+1)^2-1}{(n+1)^2}\right)$$
$$= \frac{(2-1)(2+1)}{2^2}\ \frac{(3-1)(3+1)}{3^2}\ \frac{(4-1)(4+1)}{4^2}\ \cdots\ \frac{(n+1-1)(n+1+1)}{(n+1)^2}$$
$$= \frac{1}{2}\cdot\frac{n+2}{n+1} > \frac{1}{2}.$$

6. Suppose $x_1 > x_2 > \dots$ is a decreasing sequence of real numbers. Suppose

$$x_1 + \frac{x_4}{2} + \frac{x_9}{3} + \cdots + \frac{x_{n^2}}{n} < 1$$

for all n. Show that

$$x_1 + \frac{x_2}{2} + \frac{x_3}{3} + \cdots + \frac{x_n}{n} < 3.$$

We will use these inequalities:

$$\begin{aligned} x_1, x_2, x_3 &\le x_1, \\ x_4, \cdots, x_8 &\le x_4, \\ x_9, \cdots, x_{15} &\le x_9, \\ x_{16}, \cdots, x_{24} &\le x_{16}, \end{aligned}$$

and so on. Also, we will use the inequality

$$(*) \qquad \frac{1}{n^2} + \frac{1}{n^2+1} + \cdots + \frac{1}{(n+1)^2-1} < \frac{3}{n}.$$

This is true because there are $2n+1$ terms, all $< 1/n^2$ (except one term equal to $1/n^2$) and

$$\frac{2n+1}{n^2} = \frac{2}{n} + \frac{1}{n^2} \le \frac{3}{n}.$$

Combining these inequalities, we get

$$\begin{aligned} &x_1 + \frac{x_2}{2} + \cdots + \frac{x_n}{n} \\ &\quad < \left(x_1 + \frac{x_1}{2} + \frac{x_1}{3}\right) + \left(\frac{x_4}{4} + \frac{x_4}{5} + \cdots + \frac{x_4}{8}\right) \\ &\qquad + \cdots + x_{k^2}\left(\frac{1}{k^2} + \frac{1}{k^2+1} + \cdots + \frac{1}{(k+1)^2-1}\right) + \cdots \\ &\quad < x_1(3) + x_2\left(\frac{3}{2}\right) + \cdots + x_k\left(\frac{3}{k}\right) + \cdots \le 3. \end{aligned}$$