Solutions to 2009 Rasor-Bareis Exam

1. *Find all prime numbers among 101, 10101, 1010101, 101010101, ...*

Solution. Observe that 101 is a prime number. We will show that no other number of this form is prime. The k-th number in the sequence 101, 10101, 1010101, 101010101, ... is

$$a_k = 1 + 100 + 100^2 + \ldots + 100^k = \frac{100^{k+1} - 1}{100 - 1} = \frac{(10^{k+1})^2 - 1}{99} = \frac{(10^{k+1} - 1)(10^{k+1} + 1)}{99}.$$

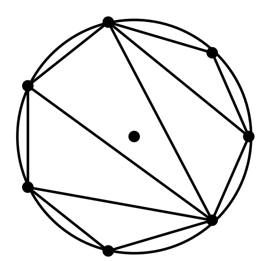
For any $k \geq 2$, both of the numbers $10^{k+1} - 1$ and $10^{k+1} + 1$ are strictly larger than 99, thus neither of these numbers is completely cancelled in the division by 99; hence, a_k is a composite number.

2. *Is there a differentiable function f on $(0, \infty)$ satisfying $f'(x) = f(x+1)$ for all x and such that $\lim_{x\to\infty} f(x) = \infty$?*

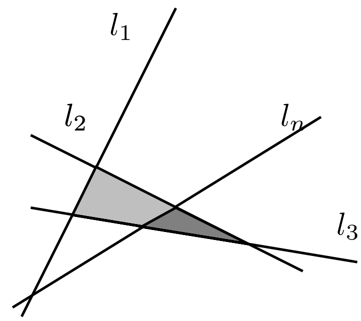
Solution. No, such a function cannot exist. Assume that $f'(x) = f(x+1)$ for all x and $\lim_{x\to\infty} f(x) = +\infty$ (if $\lim_{x\to\infty} f(x) = -\infty$, replace f by $-f$). Then for x large enough f is positive, so f' is positive, so f is increasing, so f' is increasing. But for any x, by the mean value theorem, $f(x+1) = f(x) + f'(c)$ for some $c \in [x, x+1]$, so, $f'(x) = f(x) + f'(c) > f'(c)$ if $f(x) > 0$, so f' cannot be increasing.

3. *A regular 2009-gon P is triangulated by diagonals, which means that several line segments are drawn, each joining two of the vertices of P, so that no two segments intersect except as at the end points, and that all the regions formed are triangles. Show that among the triangles thus obtained there is exactly one acute triangle.*

Solution. Circumscribe a circle C around P, then all the triangles partitioning P will be inscribed in C. A triangle inscribed in a circle is acute if and only if the center of the circle is inside of the triangle. Since 2009 is odd, no diagonal of P passes through the center O of C, and thus there exists exactly one triangle that contains O.



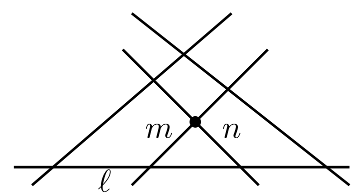
4. *Suppose $n \geq 3$ lines are drawn in the plane in general position; that is, no two of the lines are parallel, and no three of them cross at a point. Show that among the regions formed, at least one is a triangle.*



Solution. Let $l_1, \ldots, l_n$ be the lines. We proceed by induction on n. For $n = 3$, since the lines l_1, l_2, l_3 do not cross at a point and no two of them are parallel, they form a triangle.

Now assume that $n \geq 4$ and that the result is known for $n-1$ lines. So among the regions formed by the lines $l_1, \ldots, l_{n-1}$ there is a triangle, call it Δ. There are two cases to consider. If the last line l_n does not pass through the interior of Δ, then Δ is still one of the regions formed by the lines $l_1, \ldots, l_n$.

If l_n does pass through the interior of Δ, then l_n cuts from Δ a new triangle. This completes the proof by induction.



A nice student's solution. For a line ℓ, pick the intersection point of two lines that is closest to ℓ but not on ℓ. If the intersection is formed by the lines m and n, then the lines m, n and ℓ form a triangle. There can not be any line dividing up this region, because such a line would intersect either m or n at a point closer to ℓ than the intersection point of m and n, which cannot be possible because of the choice of m and n.

(Notice that this proof actually shows that for each of the n lines there is a triangle with a side on this line, and hence the number of triangles is at least $n/3$.)

5. *Find* $\sin 2\theta$ *if* $\sin^6\theta + \cos^6\theta = 2/3$.

Solution.

$$\begin{aligned}1 = (\sin^2\theta + \cos^2\theta)^3 &= \sin^6\theta + 3\sin^4\theta\cos^2\theta + 3\sin^2\theta\cos^4\theta + \cos^6\theta \\ &= \sin^6\theta + \cos^6\theta + 3\cos^2\theta\sin^2\theta(\sin^2\theta + \cos^2\theta) = \sin^6\theta + \cos^6\theta + 3\cos^2\theta\sin^2\theta,\end{aligned}$$

so

$$\cos^2\theta\sin^2\theta = \frac{1}{3}\big(1 - (\sin^6\theta + \cos^6\theta)\big) = \frac{1}{3}\Big(1 - \frac{2}{3}\Big) = \frac{1}{9}.$$

Thus, $\sin\theta\cos\theta = \pm 1/3$, and therefore $\sin 2\theta = 2\sin\theta\cos\theta = \pm 2/3$.

6. *Assume that your calculator is broken so that you can only add and subtract real numbers and compute their reciprocals. How can you use it to compute products?*

Solution. First, let us observe that, given a real number x, our broken calculator allows us to compute x^2. Indeed, for $x \notin \{0, -1\}$, since $\frac{1}{x} - \frac{1}{x+1} = \frac{1}{x^2+x}$ we have

$$x^2 = (x^{-1} - (x+1)^{-1})^{-1} - x.$$

Note also that, given x, it is easy to calculate $x/2$:

$$\frac{x}{2} = (2x^{-1})^{-1} = (x^{-1} + x^{-1})^{-1}.$$

Now, to calculate xy, we can use the identity

$$xy = \left(\frac{x+y}{2}\right)^2 - \left(\frac{x-y}{2}\right)^2$$