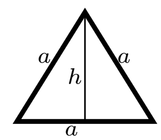
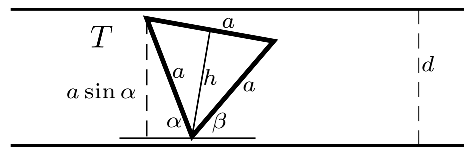
Solutions to 2010 Rasor-Bareis Prize examination problems

1. *In the plane, consider an infinite strip of width d. Suppose every triangle of area* 1 *will fit inside the strip, after suitable translation and rotation. What is the minimum possible width d?*

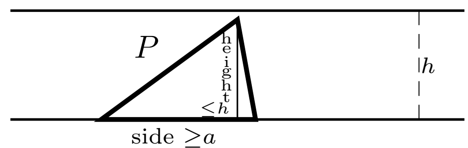
Solution. We claim that the minimum possible width d equals $\sqrt[4]{3}$, which is the height h of the equilateral triangle with area 1 (the triangle whose all sides are equal to $a = \frac{2}{\sqrt[4]{3}}$).



Indeed, if T is such a triangle that lies inside a strip of width d (see the picture), then since $\alpha + \beta + \pi/3 = \pi$, either $\alpha \geq \pi/3$ or $\beta \geq \pi/3$; if, say, $\alpha \geq \pi/3$, then $d \geq a \sin\alpha \geq a\sin(\pi/3) = h$.



On the other hand, for any triangle P of area 1, one of the sides of P has length $\geq a$. (If all sides of P have length $< a$, let γ be the minimal angle of P, so that $\gamma \leq \pi/3$; then area$(P) < \frac{1}{2}a^2 \sin\gamma \leq \frac{1}{2}a^2 \sin(\pi/3) = 1$.) The corresponding height of P is $\leq \frac{2\,\text{area}(P)}{a} = \frac{2}{2/\sqrt[4]{3}} = \sqrt[4]{3} = h$, so P can be placed inside the strip of width h as in the following picture:



2. *Given 2010 points in the plane, does there exist a straight line having 1005 points on each side of the line?*

Solution. Let A be the given set of 2010 points in the plane. Consider the set of lines containing at least two points of A. There are finitely many of them, so there is a line L that is parallel to none of these lines.

Let v be a vector orthogonal to L, and consider the family of lines parallel to L, that is, the lines of the form $L_t = L + tv$, where t is a real number. For every $a \in A$ there exists t such that $a \in L_t$, and by the choce of L, every line L_t contains at most one point of L. Let $t_1 < t_2 < \ldots < t_{2010}$ be the real numbers for which the lines L_{t_i}, $i = 1, \ldots, 2010$, contain a point of A. Then if we choose any $t \in (t_{1005}, t_{1006})$, there are exactly 1005 points of A on each side of L_t.

3. *The number 2010 is written as a sum of two or more positive integers. What is the maximum possible product of these integers?*

Solution. There are only finitely many ways to decompose 2010 into a sum of positive integers, so there is a maximum value for the product of such a decomposition. Let $a_1, \ldots, a_k$ be positive integers such that $a_1 + \ldots + a_k = 2010$ and the product $P = \prod_{i=1}^{2010} a_i$ is maximal. Then

(i) none of a_i is 1, since if $a_i = 1$ for some i then we can replace the pair a_1, a_i by the singleton $a_1 + 1$, and thereby increase the product P;
(ii) none of a_i is greater or equal than 5, since if $a_i = 5$, we can replace a_i by the pair $a_i - 2, 2$ and increase P;
(iii) moreover, we can assume no a_i is equal to 4, since 4 can be replaced by the pair $2, 2$ without changing P;
(iv) at most two of a_i are equal to 2, since otherwise we can change $2, 2, 2$ to $3, 3$ and increase P.
So, the only possible combinations for which P is maximal are 3,3,3,...,3, or 2,3,3,...,3, or 2,2,3,...,3. But since 2010 is divisible by 3, the last 2 solutions do not come up, and the maximum possible product is 3^{670}.

4. *Solve in integers the equation $x^2 + y^2 + z^2 = 2xyz$.*

Solution. Consider the family of equations

$$x^2 + y^2 + z^2 = 2^k xyz \tag{E$_k$}$$

for $k = 1, 2, \ldots$. We claim that for any k, the equation (E$_k$) has in integers only zero solution $x = y = z = 0$. Indeed, assume that there are nonzero integer solutions for some of these equations, and let (x, y, z) be "the minimal" such solution, that is, a nonzero solution of (E$_k$) for some $k \geq 1$ for which $|x| + |y| + |z|$ is minimal. Then since $2^k xyz$ is even, $x^2 + y^2 + z^2$ is also even, so either x, y and z are all even, or two of them are odd and the third is even. But in the second case $x^2 + y^2 + z^2 \equiv 2 \pmod 4$ (has remainder 2 when divided by 4) whereas $2^k xyz$ is divisible by 4, so the second case is impossible, and it must be that x, y and z are all even. Let $x = 2x_1$, $y = 2y_1$ and $z = 2z_1$; then $x_1^2 + y_1^2 + z_1^2 = \frac{1}{4} 2^k 8 x_1 y_1 z_1 = 2^{k+1} x_1 y_1 z_1$, thus (x_1, y_1, z_1) is a nonzero integer solution of the equation (E$_{k+1}$), which contradicts our assumption that (x, y, z) is the minimal solution for all k.

5. *Each vertex of a regular 2010-gon is assigned a positive real number. Suppose each of these numbers is either the arithmetic mean or the geometric mean of its two neighbors. Prove that all these numbers are equal to each other.*

Solution. Let $\mathrm{mean}(a, b)$ denote either of $\frac{a+b}{2}$ and $\sqrt{ab}$; it is only important that $a \leq \mathrm{mean}(a, b) \leq b$ if $a \leq b$, and any of $\mathrm{mean}(a, b) = a$ or $\mathrm{mean}(a, b) = b$ implies that $a = b$.

Let $a_1, a_2, \ldots, a_{2010}$ be the numbers assigned to successive vertices. Assume that $a_1 \leq a_2$ (the case $a_1 \geq a_2$ can be treated similarly.) Then since $a_2 = \mathrm{mean}(a_1, a_3)$, we get that $a_2 \leq a_3$. By induction,

$$a_1 \leq a_2 \leq a_3 \leq \ldots \leq a_{2010} \leq a_1,$$

which implies that $a_1 = a_2 = \ldots = a_{2010}$.

Another solution. For each vertex v of the 2010-gon let $a(v)$ be the number assigned to v. Suppose v is a vertex for which $a(v)$ is maximal; then for the neighboring vertices u and

w, since $a(v) = \text{mean}(a(u), a(w))$, it must be that $a(u) = a(v) = a(w)$. This means that the set of vertices v where the maximum of $a(v)$ is attained is closed under the operation of taking the neighbor. So it consists of all 2010 points.

6. *Consider an increasing sequence of positive integers*

$$1 = a_1 < a_2 < a_3 < a_4 < \dots$$

with $a_1 = 1$ *and* $a_n \leq 2a_{n-1}$ *for all* $n \geq 2$. *Prove that any positive integer not belonging to this sequence is representable as a sum of two or more distinct elements from the sequence.*

Solution. The proof is by complete induction. Suppose that we know that, for some n, every integer $b < a_n$ either is equal to one of a_k with $k < n$, or is a sum of several distinct a_k with $k < n$. Let b be an integer with $a_n < b < a_{n+1}$. Then $b - a_n < a_{n+1} - a_n \leq a_n$, so by our assumption, either $b - a_n = a_k$ for some $k < n$, in which case $b = a_k + a_n$; or $b - a_n = a_{k_1} + \ldots + a_{k_l}$ for some distinct $k_1, \ldots, k_l < n$, in which case $b = a_n + a_{k_1} + \ldots + a_{k_l}$. This proves the induction step.