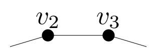
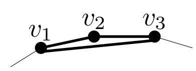
Solutions to 2011 Rasor-Bareis Prize examination problems

1. *The vertices of a regular 2011-gon are colored in two colors. Prove that some three vertices of the same color form an isosceles triangle.*

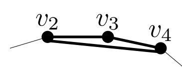
<u>*Solution*</u>. The argument we suggest solves this problem with a regular n-gon for any odd $n \geq 5$. Let "the colors" the vertices are colored are black and white, B and W. Let us denote the vertices of the n-gon by v_i, $i = 1, \ldots, n$, and write $v_i = B$ or $v_i = W$ if the vertex v_i is colored black or, respectively, white. Since n is odd, there are two neighboring vertices colored the same color; let the vertices be v_2 and v_3 and the color be B:



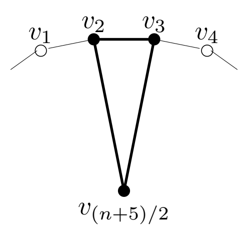
Now, if $v_1 = B$ we have a monochromatic isosceles triangle $\triangle v_1 v_2 v_3$:



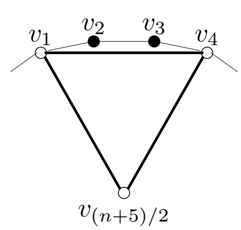
If $v_4 = B$ we have a monochromatic isosceles triangle $\triangle v_2 v_3 v_4$:



Thus, let us assume that $v_1 = v_4 = W$. But then, if $v_{(n+5)/2} = B$ the triangle $\triangle v_2 v_{(n+5)/2} v_3$ is isosceles:



and if $v_{(n+5)/2} = W$ the triangle $\triangle v_1 v_{(n+5)/2} v_4$ is isosceles:

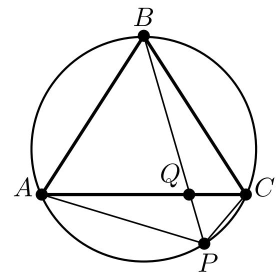


2. *Let* $g(x) = a_0 + a_1 x^{r_1} + a_2 x^{r_2} + \ldots + a_n x^{r_n}$, $x > 0$, *where* $n \in \mathbb{N}$, $a_0, a_1, \ldots, a_n$ *are nonzero real numbers and* $r_1, \ldots, r_n$ *are distinct nonzero real numbers. Prove that* g *has at most* n *zeroes in* $(0, \infty)$.

<u>*Solution*</u>. We will use induction on $n \geq 0$; for $n = 0$ the assertion is clearly true.

We have $g'(x) = a_1 r_1 x^{r_1 - 1} + a_2 r_2 x^{r_2 - 1} + \ldots + a_n r_n x^{r_n - 1}$. If $g' = 0$ then g is a nonzero constant, so it has no zeroes at all; assume that $g' \neq 0$. Thus, after collecting simular terms, g' takes the form $g'(x) = b_0 x^{s_0} + b_1 x^{s_1} + \ldots + b_m x^{s_m}$ for some $0 \leq m < n$, nonzero b_i and distinct s_i. Let $f(x) = x^{-s_0} g'(x) = b_0 + b_1 x^{p_1} + \ldots + a_n x^{p_m}$, where $p_i = s_i - s_1$, $i = 1, \ldots, m$; then f has the same zeroes on $(0, \infty)$ as g'. By (complete) induction, f has at most m zeroes, so g' has at most $m \leq n - 1$ zeroes. But then g has at most n zeroes, since, by Rolle's theorem, between any two roots of g there is a root of g'.

3. *Let ABC be an equilateral triangle and P be a point on the arc AC of the circumscribed circle. Prove that $|BP| = |AP| + |CP|$.*



<u>*Solution*</u>. Let $a = |AB|$ $(= |BC| = |AC|)$. Let Q be the point of intersection of the intervals AC and BP. The triangles ABQ and ABP are similar; indeed, they have a common angle at the vertex B and the angles $\angle BAQ$ and $\angle BPA$ are equal (both are equal to $\pi/3$) since they are subtended by equal chords.

It follows that

$$|AP|/|AQ| = |BP|/|AB|,$$

so

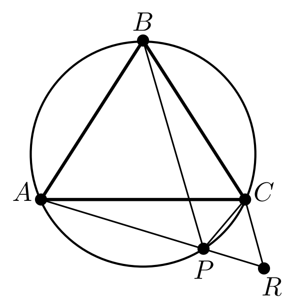
$$|AP| = |BP| \cdot |AQ|/a.$$

Similarly (from the similarity of the triangles CBQ and CBP),

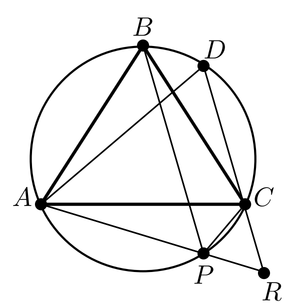
$$|CP| = |BP| \cdot |CQ|/a.$$

Hence,

$$|AP| + |CP| = |BP| \cdot (|AQ| + |CQ|)/a = |BP| \cdot a/a = |BP|.$$



<u>*Another solution*</u>. Let R be the point on the ray $[A, P)$ on the other side of P than A and such that $|PR| = |PC|$. Since $\angle ABC = \pi/3$, the angle $\angle APC = 2\pi/3$, and so $\angle CPR = \pi/3$. Hence, the triangle $\triangle CPR$ is equilateral. So, $\angle CRP = \pi/3 = \angle CPB$; since also $\angle CAP = \angle CBP$ and $|AC| = |BC|$, the triangles $\triangle BCP$ and $\triangle ACR$ are congruent (or "equal", in another terminology). So, $|BP| = |AR| = |AP| + |PR| = |AP| + |CP|$.



<u>*Yet another solution*</u>. Let D be the point on the circle between B and C such that the arc BD is equal to the arc CP. Let R be the point of intersection of the lines DC and AP. (They meet since DC is parallel to BP, and AP and BP meet.) The angle $\angle ADC = \angle ABC = \pi/3$, and the angle $\angle DAP = \angle DAC + \angle CAP = \angle DAC + \angle BAD = \angle BAC = \pi/3$, so, the triangle $\triangle ADR$ is equilateral. Thus, $|AR| = |AD|$. Also, since $|AB| = |BC|$ and the arcs BD and CP are equal, $|AD| = |BP|$. So, $|AR| = |BP|$. Next, the trianle $\triangle PRC$ is equilateral, since $\angle CRP = \angle RPC = \pi/3$, so $|PR| = |PC|$ and $|AR| = |AP| + |PC|$. Hence, $|AP| + |PC| = |BP|$.

4. *Prove that for any real numbers a, b, c,* $\quad a^4 + b^4 + c^4 \ge abc(a + b + c)$.

<u>*Solution*</u>. By the arithmetic-geometric mean inequality,

$$a^4 + b^4 + c^4 = \tfrac{1}{2}(a^4 + b^4) + \tfrac{1}{2}(b^4 + c^4) + \tfrac{1}{2}(c^4 + a^4) \ge \sqrt{a^4b^4} + \sqrt{b^4c^4} + \sqrt{c^4a^4} = a^2b^2 + b^2c^2 + c^2a^2.$$

Applying this inequality again, we get

$$a^2b^2 + b^2c^2 + c^2a^2 = \tfrac{1}{2}(a^2b^2 + b^2c^2) + \tfrac{1}{2}(b^2c^2 + c^2a^2) + \tfrac{1}{2}(c^2a^2 + a^2b^2)$$

$$\ge \sqrt{a^2b^4c^2} + \sqrt{b^2c^4a^2} + \sqrt{c^2a^4b^2} \ge ab^2c + bc^2a + ca^2b = abc(a + b + c).$$

$(\sqrt{a^2b^4c^2} > ab^2c$ if $ac < 0$ and $b \ne 0.)$

5. *Solve the equation* $\dfrac{x}{2+\dfrac{x}{2+\ddots_{\,2+\frac{x}{2+\frac{x}{1+\sqrt{1+x}}}}}}=1$*, where "2" appears 2011 times.*

<u>*Solution*</u>. Since, clearly, $x \neq 0$, we have

$$\frac{x}{1+\sqrt{1+x}}=\frac{x}{\sqrt{1+x}+1}\cdot\frac{\sqrt{1+x}-1}{\sqrt{1+x}-1}=x\cdot\frac{\sqrt{1+x}-1}{(1+x)-1}=\sqrt{1+x}-1,$$

and so,

$$2+\frac{x}{1+\sqrt{1+x}}=1+\sqrt{1+x}.$$

Thus, also $2+\dfrac{x}{2+\frac{x}{1+\sqrt{1+x}}}=2+\dfrac{x}{1+\sqrt{1+x}}=1+\sqrt{1+x}$, and by induction,

$$2+\frac{x}{2+\ddots_{\,2+\frac{x}{2+\frac{x}{1+\sqrt{1+x}}}}}=1+\sqrt{1+x},$$

independently of how many times the "2" appears... So, the equation we need to solve is $\frac{x}{1+\sqrt{1+x}}=1$, which is equivalent to $\sqrt{1+x}-1=1$, from which $x=3$.

6. *Let a polynomial* $p(x,y)$ *satisfy* $p(x+y,y-x)=p(x,y)$ *for all* $x,y\in\mathbb{R}$*. Prove that* p *is constant.*

<u>*Solution*</u>. For any $x,y\in\mathbb{R}$, applying the identity $p(u+v,u-v)=p(u,v)$ to $u=x+y$, $v=x-y$, we get

$$p(x-y,x+y)=p(u+v,u-v)=p(2y,-2x);$$

so, $p(x,y)=p(2y,-2x)$. Applying this new identity twice we obtain that $p(x,y)=p(-4x,-4y)$. But this implies that p is constant. Indeed, if p has a nontrivial monomial cx^ny^m, $c\neq 0$, then the corresponding monomial of $p(-4x,-4y)$ is $(-4)^{n+m}cx^ny^m\neq cx^ny^m$; but two polynomials are equal iff all their corresponding monomials are equal. (We consider this fact as well known and leave it without proof.)

Actually, the proven fact remains true if p is not a polynomial but any function continuous at 0. Indeed, let p be such a function and let $p(0,0)=a$. Since p is continuous at 0, for any $\varepsilon>0$ there exists $\delta>0$ such that $|p(x,y)-a|<\varepsilon$ whenever $|x|,|y|<\delta$. But for any x,y we can find $k\in\mathbb{N}$ such that $4^{-k}|x|,4^{-k}|y|<\delta$, and then

$$|p(x,y)-a|=\big|p((-4)^{-k}x,(-4)^{-k}y)-a\big|<\varepsilon.$$

Since this is true for any $\varepsilon>0$, we have $p(x,y)=a$ for all $x,y\in\mathbb{R}$.