Solutions to 2012 Rasor-Bareis Prize examination problems

1. *Let $n \in \mathbb{N}$ and $a \in \mathbb{R}$. Solve, in real numbers, the system of equations*

$$\begin{cases} x_1 + \ldots + x_n = a & (1) \\ x_1^2 + \ldots + x_n^2 = a^2 & (2) \\ x_1^3 + \ldots + x_n^3 = a^3 & (3) \\ \vdots \qquad\qquad \vdots & \vdots \\ x_1^n + \ldots + x_n^n = a^n & (n) \end{cases}$$

<u>*Solution*</u>. We will show that the only solutions of the system are $(a, 0, 0, \ldots, 0, 0)$, $(0, a, 0, \ldots, 0, 0)$, $\ldots$, $(0, 0, 0, \ldots, 0, a)$. If $n = 1$, we have only one equation $x_1 = a$. If $n = 2$, we have the system $\begin{cases} x_1 + x_2 = a \\ x_1^2 + x_2^2 = a^2, \end{cases}$ from which $2x_1x_2 = (x_1 + x_2)^2 - (x_1^2 + x_2^2) = a^2 - a^2 = 0$, so either $x_1 = 0$, $x_2 = a$, or $x_2 = 0$, $x_1 = a$.

Let $n \geq 3$. From equation (2), either $x_i^2 = a^2$ for some i and $x_j = 0$ for all $j \neq i$, or $|x_i| < |a|$ for all i. But if $|x_i| < |a|$ for all i, then

$$|x_1^3 + \ldots + x_n^3| \leq x_1^2|x_1| + \ldots + x_n^2|x_n| < (x_1^2 + \ldots + x_n^2)|a| = |a|^3,$$

which contradicts equation (3). Thus, the first possibility may only take place, that is, $x_i^2 = a^2$ for some i and $x_j = 0$ for all $j \neq i$. And from equation (1), $x_i = a$.

2. *Let f be a real-valued function on $[0, 1]$ such that $f(0) = f(1) = 0$ and $f\big(\frac{x+y}{2}\big) \leq f(x) + f(y)$ for all $x, y \in [0, 1]$. Prove that the set of zeroes of f is dense in $[0, 1]$.*

<u>*Solution*</u>. Since the dyadic rationals (numbers of the form $\frac{k}{2^n}$, $n \in \mathbb{N}$, $k \in \mathbb{Z}$) are dense in $\mathbb{R}$, it suffices to show that $f(r) = 0$ for any dyadic rational $r \in [0, 1]$. For any $x \in [0, 1]$ we have $f(x) = f\big(\frac{x+x}{2}\big) \leq f(x) + f(x)$, so $f(x) \geq 0$. Now, if for some $x, y \in [0, 1]$ we have $f(x) = f(y) = 0$, then $f\big(\frac{x+y}{2}\big) \leq f(x) + f(y) = 0$, so $f\big(\frac{x+y}{2}\big) = 0$. Thus, $f(0) = f(1) = 0$ implies that $f\big(\frac{1}{2}\big) = 0$, then $f\big(\frac{1}{4}\big) = 0$ and $f\big(\frac{3}{4}\big) = 0$, and by induction on n, $f\big(\frac{k}{2^n}\big) = 0$ for all $n \in \mathbb{N}$ and all integer k with $0 \leq k \leq 2^n$.

3. *Prove that for any $x \in \mathbb{R}$, $\sin(\cos x) < \cos(\sin x)$.*

<u>*Solution*</u>. By periodicity of sin and cos, it suffices to prove the inequality for $x \in [-\pi, \pi]$ only.

For $x = 0$ we have $\sin(\cos 0) = \sin 1 < 1 = \cos(\sin 0)$.

For any $y > 0$, $\sin y < y$; so, for any $x \in (0, \pi/2)$, since $\cos x > 0$, we have $\sin(\cos x) < \cos x$. Also, for any $x \in (0, \pi/2)$, $0 < \sin x < x < \pi/2$ and cos is a strictly decreasing function on $[0, \pi/2]$, so $\cos x < \cos(\sin x)$. Hence, $\sin(\cos x) < \cos x < \cos(\sin x)$.

For $x \in [\pi/2, \pi]$, $-1 \leq \cos x \leq 0$, so $\sin(\cos x) \leq 0$, whereas $0 \leq \sin x \leq 1$, so $\cos(\sin x) \geq \cos 1 > 0$. So, $\sin(\cos x) < \cos(\sin x)$.

For any $x \in [-\pi, 0]$ we have $\sin(\cos x) = \sin(\cos(-x)) < \cos(\sin(-x)) = \cos(-\sin x) = \cos(\sin x)$.

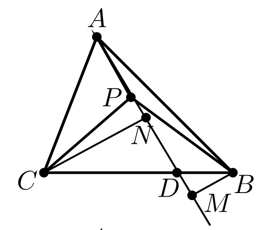
So, $\sin(\cos x) < \cos(\sin x)$ for all $x \in [-\pi, \pi]$.

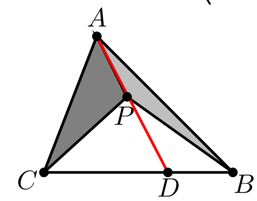
<u>*Another solution*</u>. At $x = 0$, $\sin(\cos x) = \sin 1 < 1 = \cos(\sin x)$. Both $\sin(\cos x)$ and $\cos(\sin x)$ are continuous functions, so if there exists x such that $\sin(\cos x) \geq \cos(\sin x)$, then by the intermediate value theorem there exists x such that $\sin(\cos x) = \cos(\sin x)$. For this x we have $\cos\big(\frac{\pi}{2} - \cos x\big) = \cos(\sin x)$, so $\frac{\pi}{2} - \cos x = \pm \sin x + 2n\pi$ for some $n \in \mathbb{Z}$, so $\cos x \pm \sin x = \frac{\pi}{2} - 2n\pi$ for some $n \in \mathbb{Z}$. But for any $n \in \mathbb{Z}$ and for all $x \in \mathbb{R}$,

$$\left|\frac{\pi}{2} - 2n\pi\right| \geq \frac{\pi}{2} > \sqrt{2} \geq |\cos x \pm \sin x|,$$

contradiction.

4. *Given a triangle $\triangle ABC$, find the set of points P inside this triangle such that* $\operatorname{area}(\triangle APC) = 2\operatorname{area}(\triangle APB)$.

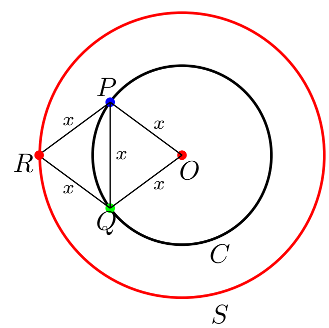




Solution. Let P be a point inside $\triangle ABC$, and let D be the point of intersection of the line (AP) with the side BC of the triangle; we will prove that $\operatorname{area}(\triangle APC)/\operatorname{area}(\triangle APB) = |CD|/|BD|$. Let M and N be the feet of the perpendiculars dropped from the vertices B and C to (AP). We have $\operatorname{area}(\triangle APC) = |AP| \cdot |CN|$ and $\operatorname{area}(\triangle APB) = |AP| \cdot |BM|$, so $\operatorname{area}(\triangle APC)/\operatorname{area}(\triangle APB) = |CN|/|BM|$. But the triangles $\triangle CND$ and $\triangle BMD$ are similar (since their sides are parallel), so $|CN|/|BM| = |CD|/|BD|$.

Hence, $\operatorname{area}(\triangle APC) = 2\operatorname{area}(\triangle APB)$ iff $|CD| = 2|BD|$. But there exists only one such point D on the side BC; hence, the points P satisfying the condition form the interval AD, where D is the point on the side BC for which $|CD| = 2|BD|$.

5. *Every point of the plane is colored one of three colors, red, blue, or green. Prove that for any $x > 0$ there are points P and Q in the plane having the same color and such that $d(P,Q) = x$, where $d(P,Q)$ denotes the distance between P and Q.*



Solution. Assume that for some $x > 0$ there are no points P, Q of the same color with $d(P,Q) = x$. Take any point O in the plane; wlog, assume that it is red. Then all points on the circle C of radius x centered at O are either blue or green. Take any point P on C; wlog, let P be blue. Then the point Q on C with $d(P,Q) = x$ must be green. Hence, the point R outside of C with $d(R,P) = d(R,Q) = x$ is red. When P runs the circle C, the corresponding point R runs the circle S of radius $\sqrt{3}x$ centered at O; so, all points of S are red. However, there are points on S with distance x between them, contradiction.

6. *Find all $n \in \mathbb{N}$ such that $p = \left\lfloor \frac{n^2}{3} \right\rfloor$ is prime.*

Solution. We will show that $n = 3$ (for which $p = 3$) and $n = 4$ (for which $p = 5$) are the only solutions.

First, note that $n = 1$ and $n = 2$ don't fit.

For $n \geq 3$ we have 3 possibilities: $n = 3k$, $n = 3k+1$, and $n = 3k+2$ for some $k \in \mathbb{N}$.

If $n = 3k$, then $p = \left\lfloor \frac{9k^2}{3} \right\rfloor = 3k^2$, is divisible by k, and is prime only if $k = 1$.

If $n = 3k+1$, then $p = \left\lfloor \frac{9k^2+6k+1}{3} \right\rfloor = 3k^2 + 2k$, is divisible by k, and is prime only if $k = 1$.

If $n = 3k+2$, then $p = \left\lfloor \frac{9k^2+12k+4}{3} \right\rfloor = 3k^2 + 4k + 1 = (3k+1)(k+1)$, and is not prime for any k.

Hence, $n = 3$ and $n = 4$ are the only solutions.