2014 Rasor-Bareis problem solutions

1. *Prove that there does not exist a prime integer of the form* 1001001…1001.

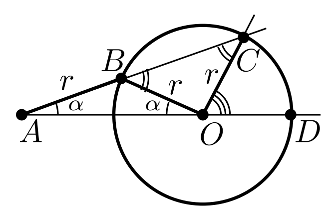
Solution. The number 1001001…1001 having n digits "1" is

$$1 + 1000 + \ldots + 1000^{n-1} = \frac{1000^n - 1}{1000 - 1} = \frac{(10^n - 1)(100^n + 10^n + 1)}{999}.$$

This number can be prime only if one of the factors in the numerator is canceled by the denominator, which is only possible if the smaller of them, $10^n - 1$, does not exceed 999, that is, if $n \le 3$. However, the integers 1, 1001, and 1001001 are not prime. (1001 is divisible by 7, and 1001001 is divisible by 3.)

2. *Suppose $|AB| = r$, the radius of the circle. If $\angle OAB = \alpha$, find $\angle DOC$.*

Solution. The triangle $\triangle ABO$ is isosceles, thus $\angle BOA = \angle BAO = \alpha$. The angle CBO is an exterior angle of the triangle $\triangle ABO$, so $\angle CBO = \angle BAO + \angle BOA = 2\alpha$. The triangle $\triangle BOC$ is also isosceles, so $\angle BCO = \angle CBO = 2\alpha$. Next, $\angle BOC = \pi - (\angle BCO + \angle CBO) = \pi - 4\alpha$. Finally, $\angle DOC = \pi - (\angle BOA + \angle BOC) = \pi - (\alpha + \pi - 4\alpha) = 3\alpha$.



3. *Let a, b, and c be the lengths of the three sides of a triangle. Prove that*

$$\frac{a}{b+c-a} + \frac{b}{a+c-b} + \frac{c}{a+b-c} \ge 3.$$

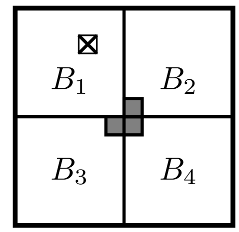
Solution. Put $x = b + c - a$, $y = a + c - b$, and $z = a + b - c$; by the triangle inequality, $x, y, z > 0$. We have $a = (y + z)/2$, $b = (x + z)/2$, and $c = (x + y)/2$, so

$$\frac{a}{b+c-a} + \frac{b}{a+c-b} + \frac{c}{a+b-c} = \frac{1}{2}\Big(\frac{y+z}{x} + \frac{x+z}{y} + \frac{x+y}{z}\Big)$$
$$= \frac{1}{2}\Big(\frac{y}{x} + \frac{x}{y}\Big) + \frac{1}{2}\Big(\frac{z}{x} + \frac{x}{z}\Big) + \frac{1}{2}\Big(\frac{z}{y} + \frac{y}{z}\Big) \ge 1 + 1 + 1 = 3,$$

by the arithmetic-geometric mean inequality.

4. *A cell is removed from a $2^n \times 2^n$ chessboard. Prove that the remaining part of the board can be tiled by L-shapes.*

<u>*Solution*</u>. For $n = 1$ the statement is obvious. Subdivide the board into four equal "sub-boards" B_1, B_2, B_3, B_4 of size $2^{n-1} \times 2^{n-1}$, as on the picture. Assume, w.l.o.g., that the cell was removed from the sub-board B_1, and remove the cells of the sub-boards B_2, B_3 and B_4 covered by an L-shape at the center of the board, as on the picture. Now exactly one square is removed from each sub-board. By induction on n, the remaining parts of the sub-boards B_1, B_2, B_3, B_4 can now be tiled by L-shapes.



5. *For any seven real numbers $y_1, \dots, y_7$ prove that there are two of them, y_i and y_j with $i \neq j$, satisfying $\left|\dfrac{y_i - y_j}{1 + y_i y_j}\right| < \dfrac{1}{\sqrt{3}}$.*

<u>*Solution*</u>. Let $x_i = \arctan y_i$, $i = 1, \dots, 7$. Since $x_1, \dots, x_7 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, for some distinct i, j we have $0 < x_i - x_j < \pi/6$. Since tan is a strictly increasing function, we then have

$$0 < \frac{y_i - y_j}{1 + y_i y_j} = \frac{\tan x_i - \tan x_j}{1 + \tan x_i \tan x_j} = \tan(x_i - x_j) < \tan(\pi/6) = \frac{1}{\sqrt{3}}.$$

6. *A rectangular pool table $ACDF$ has size 2024×1111. This table has six pockets located at the corners and at the midpoints of the longer sides, as indicated by A, B, C, D, E, F in the figure. A ball is hit from corner A at a $45°$ angle, and bounces off the edges until it falls into one of the pockets. Which pocket will it end in?*

<u>*Answer*</u>. The pocket B.

<u>*Solution*</u>. As often in billiards problems, instead of reflecting the trajectory of the ball at the points it reaches the sides of the table, let us reflect the table, keeping the trajectory to be a straight line, the line $l = \{(x, y) : y = x\}$. We obtain a lattice of points – of "pockets" – in the plane, marked by letters A, B, C, D, E, F as on the picture, with coordinates $(n|AB|, k|AC|)$ with $n, k \in \mathbb{Z}$, and now the question is what is the mark of the first point P of the lattice located on l.

(The line l intersects "the sides of the table" – the horizontal and vertical lines on the picture – at points with integer coordinates, so either the ball arrives at the center of a pocket, or hits a side of the table at a distance at least 1 from the nearest pocket.) The first pocket P on l is at the point (m, m), where m is the least common multiple of $|AB|$ and $|CD|$. We have $|AB| = 1012 = 2^2 \cdot 11 \cdot 23 = 11 \cdot 92$ and $|AF| = 1111 = 11 \cdot 101$, so $m = 101|AB| = 92|AF|$, and P has coordinates $\big(101|AB|, 92|AF|\big)$. In the "horizontal" sequence $(A)BCBABCBA\ldots$, the 101st letter is B, and in the "vertical" sequence $(B)EBEBEBE\ldots$, the 92nd letter is B. So, P is marked by "B".