2017 Rasor-Bareis exam solutions

1. *Let $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, ... be the Fibonacci numbers. Show that F_n divides F_{2n} for all $n \in \mathbb{N}$.*

Solution. Let $n \in \mathbb{N}$. We have $F_{n+2} = F_n + F_{n+1}$, and iterating this formula we get $F_{n+3} = F_{n+2} + F_{n+1} = F_n + 2F_{n+1}$, $F_{n+4} = F_{n+3} + F_{n+2} = 2F_n + 3F_{n+1}$, ... We see that the coefficients in these identities form the Fibonacci sequence themselves, so we obtain the nice identity

$$F_{n+k} = F_{k-1}F_n + F_kF_{n+1} \text{ for all } k.$$

Indeed, by induction, if this identity is true for some k and $k+1$, then for $k+2$ we also have

$$F_{n+k+2} = F_{n+k+1} + F_{n+k} = F_kF_n + F_{k+1}F_{n+1} + F_{k-1}F_n + F_kF_{n+1} = F_{k+1}F_n + F_{k+2}F_{n+1}.$$

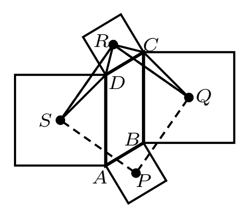
In particular, we have $F_{2n} = F_{n-1}F_n + F_nF_{n+1}$, and is divisible by F_n.

Another solution. The origin of the formulas in the first solution becomes clearer if we use a matrix representation of Fibonacci numbers. For any n, the vector $\binom{F_n}{F_{n+1}}$ is obtained from the vector $\binom{F_{n-1}}{F_n}$ by the formula $\binom{F_n}{F_{n+1}} = \binom{F_n}{F_{n-1}+F_n} = A\binom{F_{n-1}}{F_n}$, where A is the matrix $\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$. Thus, for any $n \in \mathbb{N}$, $\binom{F_{n-1}}{F_n} = A^{n-1}\binom{F_0}{F_1} = A^{n-1}\binom{0}{1}$ and $\binom{F_n}{F_{n+1}} = A^n\binom{F_1}{F_2} = A^{n-1}\binom{1}{1}$, so $A^n = \begin{pmatrix}F_{n-1} & F_n\\ F_n & F_{n+1}\end{pmatrix}$. Thus for any $n, k \in \mathbb{N}$,

$$A^{n+k} = \begin{pmatrix}F_{n+k-1} & F_{n+k}\\ F_{n+k} & F_{n+k+1}\end{pmatrix} = A^nA^k = \begin{pmatrix}F_{n-1} & F_n\\ F_n & F_{n+1}\end{pmatrix}\begin{pmatrix}F_{k-1} & F_k\\ F_k & F_{k+1}\end{pmatrix}.$$

In particular, $F_{n+k} = F_nF_{k-1} + F_{n+1}F_k$.

2. *Let $ABCD$ be a parallelogram, and let P, Q, R, S be the centers of the squares constructed on its sides. Prove that $PQRS$ is also a square.*



Solution. The triangles RDS and RCQ are coungruent: $|RD| = |RC|$, $|DS| = |CQ|$, and $\angle RDS = \angle RCQ$. Hence, $|RS| = |RQ|$ and $\angle DRS = \angle CRQ$. Since $\angle DRC = 90°$, we obtain that $\angle SRQ = 90°$ as well. Similarly, $|RQ| = |QP| = |PS|$ and $\angle RQP = \angle QPS = \angle PSR = 90°$.

3. *Prove that for each positive integer n there is a polynomial p_n of degree n with integer coefficients such that $2\cos(nx) = p_n(2\cos x)$ for all x.*

Solution. We'll use induction on n; the case $n = 1$ is trivial. For any integer n we have

$$\cos((n+1)x) + \cos((n-1)x) = 2\cos(nx)\cos x,$$

so

$$\cos((n+1)x) = 2\cos(nx)\cos x - \cos((n-1)x).$$

If by the induction hypothesis $2\cos(nx) = p_n(2\cos x)$ and $2\cos((n-1)x) = p_{n-1}(2\cos x)$, where p_{n-1} and p_n are polynomials of degree $n-1$ and n respectively with integer coefficients, we obtain that

$$2\cos((n+1)x) = p_n(2\cos x)\cdot 2\cos x - p_{n-1}(2\cos x) = p_{n+1}(2\cos x),$$

where $p_{n+1}(y) = p_n(y)y - p_{n-1}(y)$ is a polynomial of degree $n+1$ with integer coefficients.

Remark. The polynomials p_n are related to the so-called *Chebyshev's polynomials* $T_n(y) = \cos(n\arccos y)$, $n \in \mathbb{N}$, namely, $p_n(y) = 2T_n(y/2)$.

4. *Prove that there is an integer of the form* $111\dots111$ *divisible by* 2017.

Solution. Consider the sequence $u_1 = 1$, $u_2 = 11$, $u_3 = 111$, ... There are two distinct elements of this sequence, u_n and u_m with $m > n$, having the same remainder modulo 2017. The difference $u_m - u_n = 11\dots100\cdot 0 = u_{m-n}\cdot 10^n$ is divisible by 2017. The integers 10^n and 2017 are relatively prime (2017 is not divisible by 2 and 5; actually, 2017 is prime), so 2017 divides u_{m-n}.

Another solution. Since 2017 is prime, by the little Fermat's theorem, $10^{2016} = 1 \bmod 2017$, so $\underbrace{999\dots999}_{2016} = 10^{2016} - 1$ is divisible by 2017, so $\underbrace{111\dots111}_{2016}$ is divisible by 2017.

5. *If a set of integers is distributed around a circle, a legal move is to interchange some two neighboring integers* m *and* n *provided* $|m-n| \geq 3$. *Suppose that the integers* $1, 2, \dots, 2017$ *are positioned in order clockwise around the circle. Is there a sequence of legal moves that reverses the order, so that after those moves are made the integers* $1, 2, \dots, 2017$ *are positioned in order counterclockwise around the circle?*

Solution. No. Since the integers 1, 2, 3 cannot be interchanged with each other by any legal move, after any sequence of legal moves their order clockwisely remains $1, 2, 3$ and never becomes $3, 2, 1$.

6. *Evaluate* $I = \displaystyle\int_2^6 \frac{(10-x)^{2017}}{(10-x)^{2017} + (2+x)^{2017}}\, dx$.

Solution. After the substitution $y = 8 - x$, we have that $I = \int_2^6 \frac{(2+y)^{2017}}{(2+y)^{2017}+(10-y)^{2017}}\, dy$. Thus

$$2I = I + I = \int_2^6 \frac{(10-x)^{2017}}{(10-x)^{2017} + (2+x)^{2017}}\, dx + \int_2^6 \frac{(2+x)^{2017}}{(2+x)^{2017} + (10-x)^{2017}}\, dx$$
$$= \int_2^6 \frac{(10-x)^{2017} + (2+x)^{2017}}{(2+x)^{2017} + (10-x)^{2017}}\, dx = \int_2^6 1\, dx = 4.$$

So, $I = 2$.