2018 Rasor-Bareis exam solutions

1. **Prove that the decimal integer of the form 20182018⋯201820182019 cannot be a perfect square.**

   **Solution.** If \( n \in \mathbb{Z} \) is even, then \( n^2 \) is divisible by 4; if \( n \) is odd, \( n = 2k + 1 \), then \( n^2 = 4k^2 + 4k + 1 \) has residue 1 modulo 4. Any integer of the form 20182018⋯201820182019 has residue 3 modulo 4, hence it is not equal to \( n^2 \) for any \( n \in \mathbb{Z} \).

2. **Prove that** \( \sum_{1 \leq m < n \leq 2018} \frac{1}{mn} \) is not an integer.

   **Solution.** Among the positive integers not exceeding 2018 there are only two divisible by 3, namely, \( 3^6 = 729 \) and \( 2 \cdot 3^6 = 1458 \), since \( 3 \cdot 3^6 = 3^7 = 2187 > 2018 \). So, except for \( \frac{1}{3^6 \cdot (2 \cdot 3^6)} = \frac{1}{2 \cdot 3^{12}} \)

   all other summands in \( S = \sum_{1 \leq m < n \leq 2018} \frac{1}{mn} \) have the form \( \frac{1}{d \cdot 3^k} \) with \( k \leq 11 \) and \( d \) not divisible by 3. After combining all these summands, \( S \) can be written in the form

   \[
   S = \frac{a}{b \cdot 3^{11}} + \frac{1}{2 \cdot 3^{12}}
   \]

   where \( a, b \) are integers and \( b \) is not divisible by 3. Hence,

   \[
   S = \frac{6a + b}{2b \cdot 3^{12}},
   \]

   with the numerator \( 6a + b \) being not divisible by 3, and so, \( S \) is not an integer.

   **Another solution.** Notice that 2017 is a prime integer. In the sum \( S = \sum_{1 \leq m < n \leq 2018} \frac{1}{mn} \) isolate the fractions involving 2017 to find

   \[
   S = \frac{a}{b} + \sum_{m=1}^{2016} \frac{1}{m \cdot 2017} + \frac{1}{2017 \cdot 2018},
   \]

   where \( a \) and \( b \) are integers and \( b \) is not divisible by 2017. Multiplying this identity by 2017 and then reducing modulo 2017 we get

   \[
   \sum_{m=1}^{2016} m^{-1} + 2018^{-1} = 0 \mod 2017.
   \]

   However, modulo 2017, \( 2018^{-1} = 1^{-1} = 1 \) whereas \( \sum_{m=1}^{2016} m^{-1} = 0 \), since in this sum every residue modulo 2017 occurs and cancels with its negative.

3. **Evaluate** \( \int_0^\pi \arccot(\cos x) \, dx \).

   **Solution.** Let \( I = \int_0^\pi \arccot(\cos x) \, dx \). Making the substitution \( y = \pi - x \) we see that

   \[
   I = \int_0^\pi \arccot(-\cos y) \, dy = \int_0^\pi (\pi - \arccot(\cos y)) \, dy = \int_0^\pi \pi \, dy - \int_0^\pi \arccot(\cos y) \, dy = \pi^2 - I.
   \]

   So, \( I = \pi^2/2 \).
4. Prove that the perpendicular bisector of the line joining the feet of two altitudes of a triangle bisects the third side of the triangle.

Solution. Let \( AP \) and \( CQ \) be the altitudes of a triangle \( ABC \). Since \( \angle APC = \angle CQA = 90^\circ \), the points \( P \) and \( Q \) lie on the circle \( S \) built on the diameter \( AC \). The perpendicular bisector to the chord \( PQ \) of \( S \) passes through the center of \( S \), which is the middle point of \( AC \).

5. Let \( 0 < \alpha, \beta < \pi/2 \) and assume that \( \sin^2 \alpha + \sin^2 \beta = \sin(\alpha + \beta) \). Prove that \( \alpha + \beta = \pi/2 \).

Solution. Assume that \( \alpha + \beta < \pi/2 \). Then \( \alpha < \pi/2 - \beta \), and since \( \sin \) is increasing on the interval \([0, \pi/2] \),

\[
\sin \alpha < \sin\left(\frac{\pi}{2} - \beta\right) = \cos \beta.
\]

Similarly, \( \sin \beta < \cos \alpha \). Thus,

\[
\sin^2 \alpha + \sin^2 \beta < \sin \alpha \cos \beta + \sin \beta \cos \alpha = \sin(\alpha + \beta).
\]

Now assume that \( \alpha + \beta > \pi/2 \). Then \( \alpha > \pi/2 - \beta \), so

\[
\sin \alpha > \sin\left(\frac{\pi}{2} - \beta\right) = \cos \beta,
\]

and similarly, \( \sin \beta > \cos \alpha \). Thus,

\[
\sin^2 \alpha + \sin^2 \beta > \sin \alpha \cos \beta + \sin \beta \cos \alpha = \sin(\alpha + \beta).
\]

Hence, if \( \sin^2 \alpha + \sin^2 \beta = \sin(\alpha + \beta) \), then it must be that \( \alpha + \beta = \pi/2 \).

6. Prove that for any positive integer \( n \), \( (2n+1)^n \geq (2n)^n + (2n-1)^n \).

Solution. The inequality under question is equivalent to

\[
\left(1 + \frac{1}{2n}\right)^n \geq 1 + \left(1 - \frac{1}{2n}\right)^n,
\]

and so, to

\[
\left(1 + \frac{1}{2n}\right)^n - \left(1 - \frac{1}{2n}\right)^n \geq 1.
\]

And indeed, applying the binomial formula, we obtain

\[
\left(1 + \frac{1}{2n}\right)^n - \left(1 - \frac{1}{2n}\right)^n = 1 + \binom{n}{1}\left(\frac{1}{2n}\right) + \binom{n}{2}\left(\frac{1}{2n}\right)^2 + \binom{n}{3}\left(\frac{1}{2n}\right)^3 + \cdots + \binom{n}{n}\left(\frac{1}{2n}\right)^n
-1 + \binom{n}{1}\left(\frac{1}{2n}\right) - \binom{n}{2}\left(\frac{1}{2n}\right)^2 + \binom{n}{3}\left(\frac{1}{2n}\right)^3 - \cdots \pm \binom{n}{n}\left(\frac{1}{2n}\right)^n
= 2n\frac{1}{2n} + 2\binom{n}{3}\left(\frac{1}{2n}\right)^3 + \cdots \geq 1.
\]