2018 Rasor-Bareis exam solutions

1. *Prove that the decimal integer of the form* $20182018\ldots 201820182019$ *cannot be a perfect square.*

Solution. If $n \in \mathbb{Z}$ is even, then n^2 is divisible by 4; if n is odd, $n = 2k+1$, then $n^2 = 4k^2 + 4k + 1$ has residue 1 modulo 4. Any integer of the form $20182018\ldots 201820182019$ has residue 3 modulo 4, hence it is not equal to n^2 for any $n \in \mathbb{Z}$.

2. *Prove that* $\displaystyle\sum_{1\le m<n\le 2018} \frac{1}{mn}$ *is not an integer.*

Solution. Among the positive integers not exceeding 2018 there are only two divisible by 3^6, namely, $3^6 = 729$ and $2 \cdot 3^6 = 1458$, since $3 \cdot 3^6 = 3^7 = 2187 > 2018$. So, except for

$$\frac{1}{3^6 \cdot (2 \cdot 3^6)} = \frac{1}{2 \cdot 3^{12}},$$

all other summands in $S = \sum_{1\le m<n\le 2018} \frac{1}{mn}$ have the form $\frac{1}{d \cdot 3^k}$ with $k \le 11$ and d not divisible by 3. After combining all these summands, S can be written in the form

$$S = \frac{a}{b \cdot 3^{11}} + \frac{1}{2 \cdot 3^{12}}$$

where a, b are integers and b is not divisible by 3. Hence,

$$S = \frac{6a + b}{2b \cdot 3^{12}},$$

with the numerator $6a + b$ being not divisible by 3, and so, S is not an integer.

Another solution. Notice that 2017 is a prime integer. In the sum $S = \sum_{1\le m<n\le 2018} \frac{1}{mn}$ isolate the fractions involving 2017 to find

$$S = \frac{a}{b} + \sum_{m=1}^{2016} \frac{1}{m \cdot 2017} + \frac{1}{2017 \cdot 2018},$$

where a and b are integers and b is not divisible by 2017. Multiplying this identity by 2017 and then reducing modulo 2017 we get

$$\sum_{m=1}^{2016} m^{-1} + 2018^{-1} = 0 \bmod 2017.$$

However, modulo 2017, $2018^{-1} = 1^{-1} = 1$ whereas $\sum_{m=1}^{2016} m^{-1} = 0$, since in this sum every residue modulo 2017 occurs and cancels with its negative.

3. *Evaluate* $\displaystyle\int_0^\pi \operatorname{arccot}(\cos x)\, dx.$

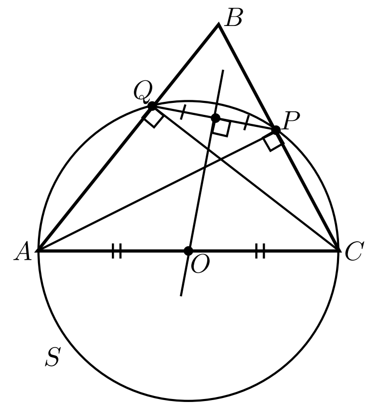
Solution. Let $I = \int_0^\pi \operatorname{arccot}(\cos x)\, dx$. Making the substitution $y = \pi - x$ we see that

$$I = \int_0^\pi \operatorname{arccot}(-\cos y)\, dy = \int_0^\pi \big(\pi - \operatorname{arccot}(\cos y)\big)\, dy = \int_0^\pi \pi\, dy - \int_0^\pi \operatorname{arccot}(\cos y)\, dy = \pi^2 - I.$$

So, $I = \pi^2/2$.

4. *Prove that the perpendicular bisector of the line joining the feet of two altitudes of a triangle bisects the third side of the triangle.*

Solution. Let AP and CQ be the altitudes of a triangle ABC. Since $\angle APC = \angle CQA = 90°$, the points P and Q lie on the circle S built on the diameter AC. The perpendicular bisector to the chord PQ of S passes through the center of S, which is the middle point of AC.



5. *Let* $0 < \alpha, \beta < \pi/2$ *and assume that* $\sin^2\alpha + \sin^2\beta = \sin(\alpha+\beta)$*. Prove that* $\alpha + \beta = \pi/2$*.*

Solution. Assume that $\alpha + \beta < \frac{\pi}{2}$. Then $\alpha < \frac{\pi}{2} - \beta$, and since sin is increasing on the interval $\left[0, \frac{\pi}{2}\right]$,

$$\sin\alpha < \sin\left(\tfrac{\pi}{2} - \beta\right) = \cos\beta.$$

Similarly, $\sin\beta < \cos\alpha$. Thus,

$$\sin^2\alpha + \sin^2\beta < \sin\alpha\cos\beta + \sin\beta\cos\alpha = \sin(\alpha+\beta).$$

Now assume that $\alpha + \beta > \frac{\pi}{2}$. Then $\alpha > \frac{\pi}{2} - \beta$, so

$$\sin\alpha > \sin\left(\tfrac{\pi}{2} - \beta\right) = \cos\beta,$$

and similarly, $\sin\beta > \cos\alpha$. Thus,

$$\sin^2\alpha + \sin^2\beta > \sin\alpha\cos\beta + \sin\beta\cos\alpha = \sin(\alpha+\beta).$$

Hence, if $\sin^2\alpha + \sin^2\beta = \sin(\alpha+\beta)$, then it must be that $\alpha + \beta = \pi/2$.

6. *Prove that for any positive integer* n*,* $(2n+1)^n \geq (2n)^n + (2n-1)^n$*.*

Solution. The inequality under question is equivalent to

$$\left(1 + \frac{1}{2n}\right)^n \geq 1 + \left(1 - \frac{1}{2n}\right)^n,$$

and so, to

$$\left(1 + \frac{1}{2n}\right)^n - \left(1 - \frac{1}{2n}\right)^n \geq 1.$$

And indeed, applying the binomial formula, we obtain

$$\begin{aligned}\left(1 + \frac{1}{2n}\right)^n - \left(1 - \frac{1}{2n}\right)^n &= 1 + \tbinom{n}{1}\left(\tfrac{1}{2n}\right) + \tbinom{n}{2}\left(\tfrac{1}{2n}\right)^2 + \tbinom{n}{3}\left(\tfrac{1}{2n}\right)^3 + \cdots + \tbinom{n}{n}\left(\tfrac{1}{2n}\right)^n \\ &\quad -1 + \tbinom{n}{1}\left(\tfrac{1}{2n}\right) - \tbinom{n}{2}\left(\tfrac{1}{2n}\right)^2 + \tbinom{n}{3}\left(\tfrac{1}{2n}\right)^3 - \cdots \pm \tbinom{n}{n}\left(\tfrac{1}{2n}\right)^n \\ &= 2n\tfrac{1}{2n} + 2\tbinom{n}{3}\left(\tfrac{1}{2n}\right)^3 + \cdots \geq 1.\end{aligned}$$