2019 Rasor-Bareis exam solutions

1. *Let $F_0 = 1$, $F_1 = 1$, $F_2 = 2$, $F_3 = 3$, ..., be the Fibonacci sequence. Prove that for every $n \in \mathbb{N}$, $\frac{1}{F_1F_3} + \frac{1}{F_2F_4} + \cdots + \frac{1}{F_nF_{n+2}} < 1$.*
Solution. For every k we have

$$\frac{1}{F_kF_{k+1}} - \frac{1}{F_{k+1}F_{k+2}} = \frac{F_{k+2} - F_k}{F_kF_{k+1}F_{k+2}} = \frac{F_{k+1}}{F_kF_{k+1}F_{k+2}} = \frac{1}{F_kF_{k+2}}.$$

Using this identity, we transform our sum into a telescoping one:

$$\frac{1}{F_1F_3} + \frac{1}{F_2F_4} + \cdots + \frac{1}{F_nF_{n+2}} = \frac{1}{F_1F_2} - \frac{1}{F_2F_3} + \frac{1}{F_2F_3} - \frac{1}{F_3F_4} + \cdots + \frac{1}{F_nF_{n+1}} - \frac{1}{F_{n+1}F_{n+2}}$$
$$= \frac{1}{F_1F_2} - \frac{1}{F_{n+1}F_{n+2}} = 1 - \frac{1}{F_{n+1}F_{n+2}} < 1.$$

2. *Suppose rational numbers a and b are such that the numbers $\sqrt[3]{a} + \sqrt[3]{b}$ and $\sqrt[3]{ab}$ are also rational. Prove that $\sqrt[3]{a}$ and $\sqrt[3]{b}$ are rational as well.*
Solution. Let $A = \sqrt[3]{a}$ and $B = \sqrt[3]{b}$; we are given that $A^3, B^3, A + B, AB \in \mathbb{Q}$ and need to show that $A, B \in \mathbb{Q}$. We have

$$A^2 + AB + B^2 = (A + B)^2 - AB \in \mathbb{Q},$$

so

$$A - B = (A^3 - B^3)/(A^2 + AB + B^2) \in \mathbb{Q},$$

so $A = (A + B)/2 + (A - B)/2 \in \mathbb{Q}$ and $B = (A + B) - A \in \mathbb{Q}$.

3. *The integer points (n, m), $n, m \in \mathbb{Z}$, of the plane $\mathbb{R}^2$ are colored with five colors so that every configuration of the form*

$$\big\{(n, m), (n - 1, m), (n + 1, m), (n, m - 1), (n, m + 1)\big\},\ \ n, m \in \mathbb{Z},$$

contains all five colors. Prove that every length 5 row

$$\big\{(n, m), (n + 1, m), (n + 2, m), (n + 3, m), (n + 4, m)\big\},\ \ n, m \in \mathbb{Z}$$

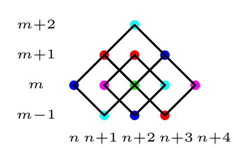
and every length 5 column

$$\big\{(n, m), (n, m + 1), (n, m + 2), (n, m + 3), (n, m + 4)\big\},\ \ n.m \in \mathbb{Z}$$

also contain all five colors.

Solution. It suffices to prove this assertion for rows only. By the way of contradiction, assume that a row

$$\big\{(n, m), (n + 1, m), (n + 2, m), (n + 3, m), (n + 4, m)\big\}$$



does not contain points of one of our five colors, say, red.
Then in each of the configurations

$$\big\{(k - 1, m), (k, m), (k + 1, m), (k, m + 1), (k, m - 1)\big\}$$

with $k = n+1, n+2, n+3$, at least one of the points $(k, m+1)$ and $(k, m-1)$ must be red. W.l.o.g. assume that in two of these three configurations the red point is $(k, m + 1)$; then the configuration

$$\big\{(n + 1, m + 1), (n + 2, m + 1), (n + 3, m + 1), (n + 2, m + 2), (n + 2, m)\big\}$$

contains two red points, which contradicts the assumption.

4. *Evaluate* $\int_0^{\pi/2} \frac{dx}{1+\tan^{2019} x}$.

Solution. Let $I = \int_0^{\pi/2} \frac{dx}{1+\tan^{2019} x}$. Since $\tan(\pi/2 - x) = \cot x$, we also have

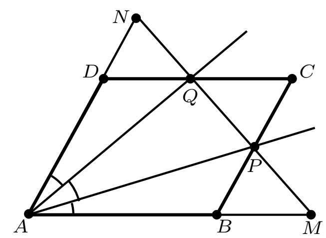
$$I = \int_0^{\pi/2} \frac{dx}{1+\tan^{2019}(\pi/2 - x)} = \int_0^{\pi/2} \frac{dx}{1+\cot^{2019} x} = \int_0^{\pi/2} \frac{\tan^{2019} x}{\tan^{2019} x + 1}\, dx,$$

and so,

$$2I = \int_0^{\pi/2} \frac{1}{1+\tan^{2019} x}\, dx + \int_0^{\pi/2} \frac{\tan^{2019} x}{1+\tan^{2019} x}\, dx = \int_0^{\pi/2} 1\, dx = \pi/2.$$

Hence, $I = \pi/4$.

5. *Let $ABCD$ be a parallelogram and let P and Q be the midpoints of BC and of DC respectively. Is it possible that the rays AP and AQ trisect the angle $\angle BAD$?*



Solution. The answer is negative. By the way of contradiction, assume that $\angle BAP = \angle PAQ = \angle QAD$. Let M be the point of intersection of the lines AB and PQ and N be the point of intersection of the lines AD and PQ. The the triangles $\triangle DQN$ and $\triangle CQP$ are congruent, so $|NQ| = |QP|$. Hence, in the triangle $\triangle ANP$, AQ is simultaneously a median and an angle bisector, thus this triangle is isosceles, so AQ is its height as well, that is, $AQ \perp PQ$. But, similarly, also $AP \perp PQ$, so $P = Q$, contradiction.

6. *Prove that for all $n, m \in \mathbb{N}$, $\frac{1}{\sqrt[m]{1+n}} + \frac{1}{\sqrt[n]{1+m}} \geq 1$.*

Solution. For any $n, m \in \mathbb{N}$, $\left(1 + \frac{n}{m}\right)^m \geq 1 + m\frac{n}{m} = 1 + n$ (Bernoulli's inequality), so $\sqrt[m]{1+n} \leq 1 + \frac{n}{m}$, and similarly, and $\sqrt[n]{1+m} \leq 1 + \frac{m}{n}$. So,

$$\frac{1}{\sqrt[m]{1+n}} + \frac{1}{\sqrt[n]{1+m}} \geq \frac{1}{1+n/m} + \frac{1}{1+m/n} = 1.$$