2020 Rasor-Bareis exam solutions

1. *Prove that* 17^{2020} *cannot be represented as* m^3+n^3 *for positive integers* m *and* n.

Solution. The assertion follows from the fact that 17^{2020} cannot be represented as a sum m^3+n^3 modulo 7. Indeed, by the little Fermat's theorem, $17^6 \equiv 1 \bmod 7$, and since $2020 = 636 \cdot 6 + 4$,

$$17^{2020} \equiv 17^4 \bmod 7 \equiv 3^4 \bmod 7 \equiv 4 \bmod 7.$$

On the other hand, for any integer m, $m^3 \bmod 7$ can only be equal to 0, 1, or $-1 \equiv 6$, and the sum of two cubes cannot be 4 modulo 7.

Another solution. Assume that for some $k \in \mathbb{N}$ there are $m, n \in \mathbb{N}$ such that $17^k = m^3+n^3$. Choose the smallest k with this property; then neither m nor n are divisible by 17. Indeed, if 17 is a factor of one of them, then it is a factor of the other, and we can cancel 17^3 to get an identity with a smaller k; but this contradicts the choice of k if $k \geq 1$ and is impossible if $k \leq 0$.

Since $m^3+n^3 = (m+n)(m^2-mn+n^2)$ and 17 is prime, the Unique Factorization theorem gives that both $m+n$ and m^2-mn+n^2 are powers of 17. Then 17 divides $(m+n)^2 = (m^2-mn+n^2)+3mn$, so, divides mn, and so divides m or n, contradiction.

2. *Prove that for any* $x, y, z \in [0,1]$, $\dfrac{x}{7+y^3+z^3} + \dfrac{y}{7+z^3+x^3} + \dfrac{z}{7+x^3+y^3} \leq \dfrac{1}{3}$.

Solution. Since $x^3, y^3, z^3 \leq 1$, we have

$$\frac{x}{7+y^3+z^3} + \frac{y}{7+z^3+x^3} + \frac{z}{7+x^3+y^3} \leq \frac{x}{6+x^3+y^3+z^3} + \frac{y}{6+x^3+y^3+z^3} + \frac{z}{6+x^3+y^3+z^3} = \frac{x+y+z}{6+x^3+y^3+z^3}.$$

To prove that this quotient is $\leq \frac{1}{3}$, we need to show that $6+x^3+y^3+z^3 \geq 3x+3y+3z$, and we are done if we have $2+x^3 \geq 3x$ for all $x \in [0,1]$. But this is so indeed since the polynomial x^3-3x+2 is decreasing on $[0,1]$ and vanishes at 1.

3. *Prove that* $\displaystyle\int_0^{\pi/2} \cos(2020x)(\cos x)^{2018}\, dx = 0$.

Solution. For any $n \in \mathbb{N}$, by a trigonometric formula, we have

$$I = \int_0^{\pi/2} \cos\big((n+2)x\big)(\cos x)^n\, dx = \int_0^{\pi/2} \cos\big((n+1)x\big)(\cos x)^{n+1}\, dx - \int_0^{\pi/2} \sin\big((n+1)x\big)\sin x\,(\cos x)^n\, dx.$$

Integrating the second term by parts, with $u = \sin\big((n+1)x\big)$ and $v = (\cos x)^n \sin x$, we get

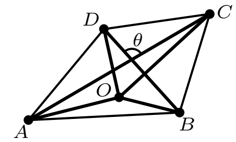
$$\int_0^{\pi/2} \sin\big((n+1)x\big)\sin x\,(\cos x)^n\, dx = -\frac{1}{n+1}\sin\big((n+1)x\big)(\cos x)^{n+1}\Big|_0^{\pi/2} + \int_0^{\pi/2} (\cos x)^{n+1}\cos\big((n+1)x\big)\, dx = \int_0^{\pi/2} \cos\big((n+1)x\big)(\cos x)^{n+1}\, dx.$$

Subtracting this from the first term, we get $I = 0$.

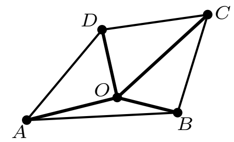
4. *Find all real polynomials* $f(x) = x^{2020} + a_{2019}x^{2019} + \cdots + a_1x + a_0$ *all of whose roots are real, and such that* $|f(i)| = 1$.

Solution. Let $x_1, \ldots, x_{2020}$ be the roots of f (listed with their multiplicities), so that $f(x) = \prod_{k=1}^{2020}(x-x_k)$. For every k, since $x_k \in \mathbb{R}$, we have $|i-x_k| = \sqrt{x_k^2+1^2} \geq 1$, with equality only if $x_k = 0$. Hence, $|f(i))| = \prod_{k=1}^{2020}|i-x_k| \geq 1$ with equality only if $x_k = 0$ for all k. Hence, all roots of f are equal to 0, and $f(x) = x^{2020}$.

5. *Let $ABCD$ be a convex quadrilateral of area 1, and let O be a point inside it. Prove that $|AO|+|BO|+|CO|+|DO| \geq 2\sqrt{2}$.*



Solution. By the triangle inequality, $|AO|+|CO| \geq |AC|$ and $|BO|+|DO| \geq |BD|$, so $|AO|+|BO|+|CO|+|DO| \geq |AC+|BD|$. The area S of $ABCD$ equals $\frac{1}{2}|AC| \cdot |BD| \sin\theta$, where θ is the angle between AC and BD. Hence $|AC| \cdot |BD| \geq 2S = 2$. By the AM-GM inequality, $|AC|+|BD| \geq 2\sqrt{|AC| \cdot |BD|} \geq 2\sqrt{2}$.

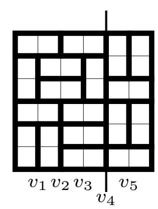


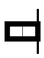
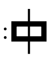
Another solution. By the formula for the area of a trinagle, the sum of the areas of the triangles AOB, BOC, COD, and DOA do not exceed $\frac{1}{2}|AO| \cdot |BO|$, $\frac{1}{2}|BO| \cdot |CO|$, $\frac{1}{2}|CO| \cdot |DO|$, and $\frac{1}{2}|DO| \cdot |AO|$ respectfully. Thus,

$$2 \leq |AO| \cdot |BO| + |BO| \cdot |CO| + |CO| \cdot |DO| + |DO| \cdot |AO| = (|AO|+|CO|)(|BO|+|DO|).$$

By the AM-GM inequality, $\sqrt{(|AO|+|CO|)(|BO|+|DO|)} \leq \frac{1}{2}\big((|AO|+|CO|)+(|BO|+|DO|)\big)$, so $|AO|+|CO|+|BO|+|DO| \geq 2\sqrt{2}$.

6. *A 6×6 board is covered with eighteen 2×1 tiles, without gaps or overlaps. No matter how those tiles are arranged, prove that there always is a straight line that cuts across the whole board without cutting any tile.*



Solution. Naturally, we only consider the lines of the grid subdividing the board into its 1×1 squares; there are 10 such lines, 5 "vertical" and 5 "horizontal", and every tile can be cut, into halfs, by at most one of these lines. We claim that each line cuts an even (possibly, zero) number of tiles. Indeed, let's enumerate the vertical lines, $v_1, \ldots, v_5$, in order, from the left to the right. For each i, let a_i be the number of "vertical" tiles for which v_i serves as the right edge: , b_i be the number of "horizontal" tiles for which v_i serves as the right edge: , and c_i be the number of (horizontal) tiles halved by v_i: . Then $2a_i + b_i + c_i = 10$ and $b_i = c_{i-1}$. Since $b_1 = 0$, we have by induction that b_i and c_i are even for all i. The same applies to the horizontal lines. It follows that if a line cuts a tile, then it cuts ≥ 2 tiles, and if every line cuts tiles then all together they cut $10 \cdot 2 \geq 20$ tiles; but this is impossible since there are only 18 those.