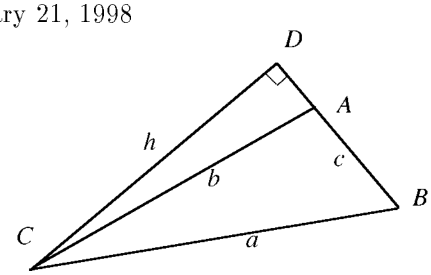
Rasor-Bareis Exam: solutions, February 21, 1998



1. Let the triangle have vertices ABC, where A is the vertex opposite to side a, and similarly for B, C. If $\angle CAB = 90°$, then the area of $\triangle ABC$ equals $bc/2 = 1$, and since $b \geq c$, we have $b = \sqrt{b^2} \geq \sqrt{bc} = \sqrt{2}$. On the other hand, if $\angle CAB \neq 90°$, let D be the point on the side AB (or its continuation, as in the picture) such that CD is perpendicular to AB. Then, if h is the length of CD, the area of $\triangle ABC$ equals $hc/2 = 1$. Notice in the right triangle CDA, that $b > h$, so we have $b = \sqrt{b^2} > \sqrt{hb} = \sqrt{2}$.

2. Let $f(x) = |\sin x|$ and $g(x) = 2x/(1997\pi)$. First note that $0 \leq f(x) \leq 1$ for all x, so the equation $f(x) = g(x)$ has no solutions outside the interval where $0 \leq g(x) \leq 1$, i.e., outside the interval $[0, 1997\pi/2]$. We will show that the equation has exactly two solutions on each of the half-open intervals $[(n-1)\pi, n\pi)$ for $n = 1, 2, 3, \ldots, 999$, so it has $2 \cdot 999 = 1998$ solutions in all.

On each open interval $((n-1)\pi, n\pi)$, we have $f''(x) = -f(x) < 0$. Hence, f is strictly concave on each closed interval $[(n-1)\pi, n\pi]$, so its graph cannot meet any line in more than two points within such an interval. So $f(x) = g(x)$ has at most two solutions there.

For $1 \leq n \leq 998$, we have $f((n-1)\pi) = 0 \leq g((n-1)\pi)$, $f((n-0.5)\pi) = 1 > g((n-0.5)\pi)$, and $f(n\pi) = 0 < g(n\pi)$, so the Intermediate Value Theorem gives at least one solution to $f(x) = g(x)$ in each of the subintervals $[(n-1)\pi, (n-0.5)\pi)$ and $((n-0.5)\pi, n\pi)$. For $n = 999$, note that $f((n-1)\pi) = 0 \leq g((n-1)\pi)$, $f((n-0.5)\pi) = 1 = g((n-0.5)\pi)$, and $f(x) > g(x)$ for x less than but sufficiently close to $(n-0.5)\pi$, because $f'((n-0.5)\pi) = 0 < g'((n-0.5)\pi)$. Hence, there must be a solution to $f(x) = g(x)$ strictly between $(n-1)\pi$ and $(n-0.5)\pi$, giving two solutions between $(n-1)\pi$ and $n\pi$.

3. See Problem 2 of the Gordon contest.

4. See Problem 3 of the Gordon contest.

5. *Solution # 1.* If 1998 is sum of n consequtive integers, i.e. $1998 = m + (m+1) + \cdots + (m+n-1)$, where $n \geq 2$, then

$$1998 = \frac{(2m+n-1)n}{2}.$$

Since $1998 = 2 \cdot 3^3 \cdot 37$, we have $2^2 \cdot 3^3 \cdot 37 = (2m+n-1)n$. Keeping in mind that m and n must be positive integers, it is easy to see that only 7 cases are possible: $n =$ 3, 4, 9, 12, 27, 36, and 37.

Solution # 2. If 1998 is a sum of n consequtive positive integers, then either n is odd, say, $n = 2l+1$, in which case $1998 = \sum_{i=-l}^{l}(a+i) = na$ for some positive integer $a > l$, or n is even, say, $n = 2l$, in which case $1998 = n(a+1/2)$ for some positive integer $a > l$. The prime number decomposition of the number 1998 is $1998 = 2 \cdot 3^3 \cdot 37$. If n is odd, then $a = 1998/n$ must be an integer bigger than $n/2$. Checking different combinations of 3 and 37 we conclude that only 4 cases are possible: $n =$ 3, 9, 27, and 37. If n is even, then $1998/n$ must be a half-integer bigger than $n/2$, therefore, n is 4 times a number that divides 1998. It is easy to see that $n =$ 4, 12, or 36. Answer: 7.

6. All of the following steps are reversible (squaring is order-preserving on nonnegative numbers): $3\sqrt{3} - 2\sqrt{6} + 7\sqrt{5} - 5\sqrt{10} > 0 \iff 3\sqrt{3} + 7\sqrt{5} > 2\sqrt{6} + 5\sqrt{10} \iff (3\sqrt{3}+7\sqrt{5})^2 > (2\sqrt{6}+5\sqrt{10})^2 \iff 27 + 42\sqrt{15} + 245 > 24 + 20\sqrt{60} + 250 \iff 42\sqrt{15} > 2 + 40\sqrt{15} \iff 2\sqrt{15} > 2 \iff \sqrt{15} > 1$. This is true, so the original inequality is true: the expression is positive.

1. *Solution # 1.* We will show by induction on $n \geq 2$ that the area of an n-gon $P_n = A_1 A_2 \cdots A_n$ inscribed in the circle with center O and radius R achieves its maximum value $((1/2)R^2 n \sin(2\pi/n))$ when the n-gon is regular. To start the induction, note that for $n = 2$, when the n-gon is just a line segment, the area is 0, so this formula is correct.

Let $\theta_i = \angle A_i O A_{i+1}$, $i = 1, 2, \cdots, n-1$ and $\theta_n = 2\pi - (\theta_1 + \theta_2 + \cdots + \theta_{n-1})$. Then the area of P_n is given by

$$S(P_n) = \frac{1}{2} R^2 \sin\theta_1 + \frac{1}{2} R^2 \sin\theta_2 + \cdots + \frac{1}{2} R^2 \sin\theta_{n-1} + \frac{1}{2} R^2 \sin\left(2\pi - (\theta_1 + \theta_2 + \cdots + \theta_{n-1})\right).$$

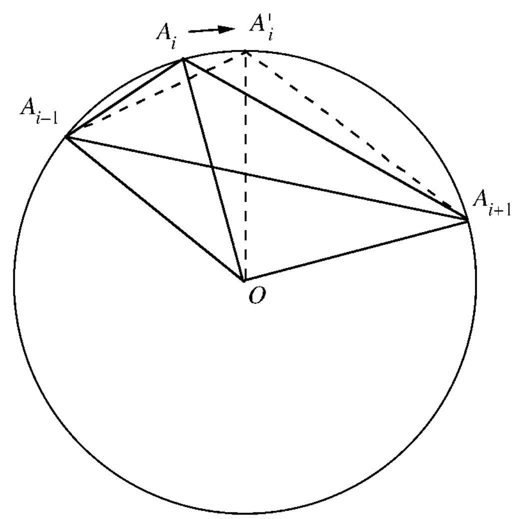
Thus the problem is reduced to finding the maximum value of the continuous function

$$f(\theta_1, \theta_2, \cdots, \theta_{n-1}) = \sin\theta_1 + \sin\theta_2 + \cdots + \sin\theta_{n-1} + \sin\left(2\pi - (\theta_1 + \theta_2 + \cdots + \theta_{n-1})\right)$$

in the region D_n defined by the inequalities $\theta_i \geq 0$, $i = 1, 2, \cdots, n-1$, and $\theta_1 + \theta_2 + \cdots + \theta_{n-1} \leq 2\pi$. Computing the partial derivatives and equating them to 0 gives us the *unique* solution in the *interior* of D_n, namely $\theta_1 = \theta_2 = \cdots = \theta_{n-1} = 2\pi/n$ [and hence $\theta_n = 2\pi/n$]. It remains to show that the corresponding value $f(\theta_1, \cdots, \theta_{n-1}) = n\sin(2\pi/n)$ is larger than the values of f on the boundary of D_n. This follows from the induction hypothesis, since on the boundary of D_n one or more of the θ_i are equal to 0, and the problem of maximization is the same problem, but with fewer variables. But since $k\sin(2\pi/k) < n\sin(2\pi/n)$ for all $k < n$, we are done.

Solution # 2. The following is an *elementary* solution, not invoking calculus.

If $P = A_1 A_2 \cdots A_n$ is an inscribed n-gon, write $m(P)$ for the number of triangles $\Delta A_i O A_{i+1}$ such that $\angle A_i O A_{i+1} = 2\pi/n$. [Write $A_{n+1} = A_1$ for convenience.] We will show that if $m(P) < n$, then we can find an inscribed n-gon $\widetilde{P}$ with larger area such that $m(\widetilde{P}) \geq m(P) + 1$. Indeed, notice that if $m(P) < n$, then $m(P)$ is at most $n-2$, and among the triangles $\Delta A_i O A_{i+1}$ there are at least two, call them T_1 and T_2, such that T_1 has central angle $< 2\pi/n$ and T_2 has central angle $> 2\pi/n$. Cutting and pasting, if needed, we may assume without loss of generality that T_1 and T_2 are adjacent, say $T_1 = \Delta A_{i-1} O A_i$ and $T_2 = \Delta A_i O A_{i+1}$. Moving the vertex A_i to a new position A_i' such that $\angle A_{i-1} O A_i' = 2\pi/n$ (and keeping all the other vertices fixed) increases the area of the n-gon and increases the value of the parameter $m(P)$. The area increases, since the sum of the areas of the two triangles T_1, T_2 is the sum of Area($\Delta A_{i-1} O A_{i+1}$) and Area($\Delta A_{i-1} A_i A_{i+1}$). The first area remains unchanged when vertex A_i is moved, but Area($\Delta A_{i-1} A_i A_{i+1}$) $<$ Area($\Delta A_{i-1} A_i' A_{i+1}$) since the triangles both have the same base $A_{i-1} A_{i+1}$, but the second one has greater height perpendicular to $A_{i-1} A_{i+1}$. (The reader should observe that a simple modification of this argument will work in the case when $\angle A_{i-1} O A_i + \angle A_i O A_{i+1}$ is greater than or equal to π.)



After finitely many steps, always increasing the area, we arrive at the situation where $m(P) = n$.

2. *Solution # 1.* Consider the function $f(x) = \sin^{1998} x + \cos^{1998} x$. It is periodic with period 2π, so its minimum value on the whole real line is the same as its minimum value on the interval $[0, 2\pi]$. The function is continuous, so it attains a minimum value on $[0, 2\pi]$. The function is differentiable, so that minimum value occurs at a critical point or an endpoint. So let us find the critical points. The derivative is $f'(x) = 1998 \sin^{1997} x \cos x - 1998 \cos^{1997} x \sin x = 1998(\sin x)(\cos x)(\sin^{1996} x - \cos^{1996} x)$. Now this derivative is zero only when one of the factors is zero; that is: (a) when $1998 = 0$ [never], or (b) when $\sin x = 0$ [so $x = 0, \pi, 2\pi$], or (c) when $\cos x = 0$ [so $x = \pi/2, 3\pi/2$], or (d) when $\sin^{1996} x = \cos^{1996} x$, so $\sin x = \pm\cos x$ [so $x = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$]. Plugging in these values, we get $f(x) = 1$ when x is any of the values $0, \pi/2, \pi, 3\pi/2, 2\pi$ and $f(x) = 1/2^{998}$ when x is any of the values $\pi/4, 3\pi/4, 5\pi/4, 7\pi/4$. So the minimum value is $1/2^{998}$, as claimed.

Solution # 2. A more elementary proof may be done using the inequality $a^n + b^n \geq (a+b)^n/2^{n-1}$ for any positive numbers a, b. (This may be proved by induction.) Applying the inequality with $a = \sin^2 x$, $b = \cos^2 x$, and $n = 999$, we get: $\sin^{1998} x + \cos^{1998} x \geq (\sin^2 x + \cos^2 x)^{999}/2^{998} = 1/2^{998}$.

3. First of all we will figure out, for a given value k, how many integers n there are with $a_n = k$. Note that $a_n = k$ if $k - (1/2) \leq \sqrt{n} < k + (1/2)$; that is, if $k^2 - k + (1/4) \leq n < k^2 + k + (1/4)$. Since n and k are integers, this is equivalent to $k^2 - k < n \leq k^2 + k$, so there are exactly $2k$ such n. It follows that for every $k \in \mathbb{N}$,

$$\sum_{n\,:\,a_n=k} \frac{1}{a_n} = \frac{2k}{k} = 2.$$

The last k such that all n with $a_n = k$ are inside the interval $\{1, 2, \cdots, 1998\}$ is 44 [since $44^2 + 44 = 1980$ and $45^2 + 45 > 1998$]. Thus we have 44 "full groups," and a part of the 45th:

$$\sum_{n=1}^{1998} \frac{1}{a_n} = \sum_{k=1}^{44} \sum_{n\,:\,a_n=k} \frac{1}{a_n} + \sum_{n=1981}^{1998} \frac{1}{a_n} = 44 \cdot 2 + \frac{18}{45} = 88.4.$$

4. If $a = 0$, then $|a| = |b| = |c|$ gives $b = c = 0$, so $a^3 = b^3 = c^3$. So we may assume $a \neq 0$. Let $B = b/a$ and $C = c/a$; then we have $|B| = |C| = 1$ and $1 + B + C = (a + b + c)/a = 0$.

Let $B = x + yi$ with x and y real; then $C = (-1 - x) - yi$. Now $|B| = |C| = 1$ gives $x^2 + y^2 = 1$ and $(1 + x)^2 + y^2 = 1$; subtracting these two equations yields $1 + 2x = 0$, so $x = -1/2$. Now we can solve $x^2 + y^2 = 1$ for y to get $y = \pm\sqrt{3}/2$, so $B = -1/2 \pm (\sqrt{3}/2)i$ and $C = -1/2 \mp (\sqrt{3}/2)i = \overline{B}$. A direct computation now gives $B^2 = C$, so $B^3 = BC = B\overline{B} = |B|^2 = 1$ and $C^3 = \overline{B^3} = 1$. Therefore, $b^3 = (Ba)^3 = a^3$ and $c^3 = (Ca)^3 = a^3$.

5. The limit is 0. It is enough to show that a_n converges to 0, where $a_n = |\sin\alpha\ \sin 2\alpha\ \cdots \sin n\alpha|$. Since the sequence a_n is nonincreasing and bounded below, it converges; call the limit a. Let us show that the assumption $a \neq 0$ leads to a contradition. Indeed, note that if $a \neq 0$, then

$$|\sin n\alpha| = \frac{a_n}{a_{n-1}} \to \frac{a}{a} = 1.$$

But since $\sin^2 n\alpha + \cos^2 n\alpha = 1$, it follows that $\cos n\alpha \to 0$. But then from the addition formula for the sine, we have

$$1 = \lim_{n\to\infty} |\sin n\alpha| = \lim_{n\to\infty} \left|\sin(n-1)\alpha\ \cos\alpha + \sin\alpha\ \cos(n-1)\alpha\right| = |1 \cdot \cos\alpha + \sin\alpha \cdot 0| = |\cos\alpha|.$$

But since $\sin^2\alpha + \cos^2\alpha = 1$, this implies that $\sin\alpha = 0$, and that means that $a_n = 0$ for all n. This obviously contradicts $\lim a_n = a \neq 0$ and we are done.

6. The limit is $1/2$. If we re-write the sum properly:

$$\sum_{k=1}^{n} \frac{n}{(k+n)^2} = \sum_{k=1}^{n} \frac{1}{\left(1 + \dfrac{k}{n}\right)^2} \cdot \frac{1}{n},$$

then we may recognize it as a Riemann sum for the integral $\int_1^2 (1/x^2)\,dx$. The integrand $1/x^2$ is continuous on the interval $[1, 2]$, and the norm $1/n$ of our partitions goes to 0 as $n \to \infty$, so these Riemann sums converge to the value of this integral, $1/2$.