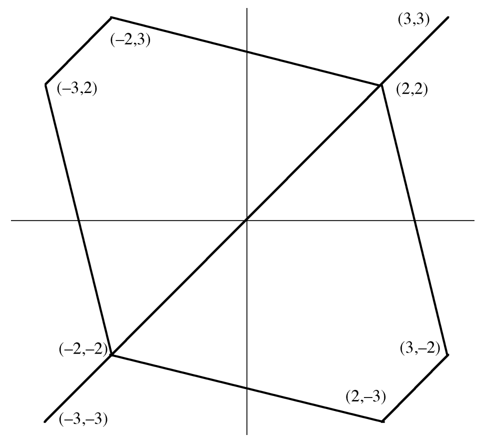
Rasor-Bareis Exam: solutions, February 27, 1999

1. *See Problem 1 of the Gordon Exam.*

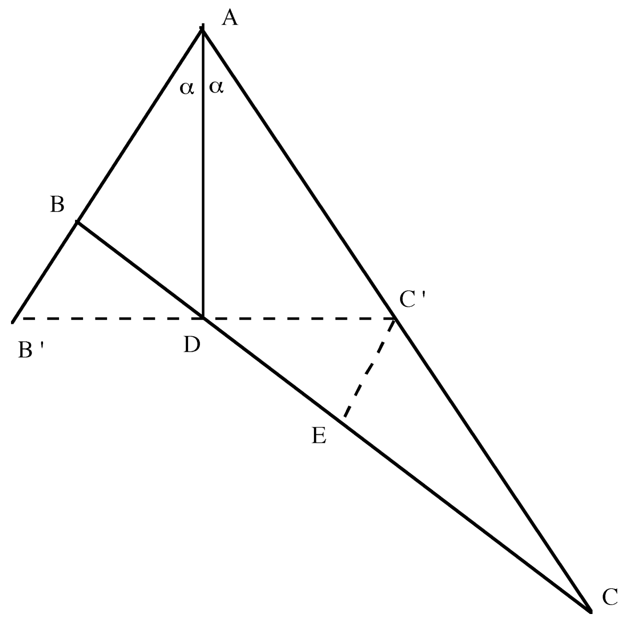
2. Let us write the three line segments parametrically. They are:

$$T_1 = \{\, ((t-5)/2, t) : -1 \le t \le 1 \,\},$$
$$T_2 = \{\, (-2t, t) : -1 \le t \le 1 \,\}, \text{ and}$$
$$T_3 = \{\, ((t+5)/2, t) : -1 \le t \le 1 \,\}.$$



So the graph of $f(x) = f(y)$ consists of: (1) all pairs (x, y) with $x = y$ between -3 and 3, that is the line segment from $(-3, -3)$ to $(3, 3)$; (2) all pairs (x, y) where for some t, $(x, t) \in T_1$ and $(y, t) \in T_2$, that is the set of pairs $\{\, ((t-5)/2), -2t) : -1 \le t \le 1 \,\}$, or the line segment from $(-3, 2)$ to $(-2, -2)$; (3) (x, y) where $(x, t) \in T_2$ and $(y, t) \in T_1$, which gives us the line segment from $(2, -3)$ to $(-2, -2)$; (4) T_1 and T_3 yield the segment from $(-3, 2)$ to $(-2, 3)$; (5) T_3 and T_1 yield $(2, -3)$ to $(3, -2)$; (6) T_2 and T_3 yield $(2, 2)$ to $(-2, 3)$; (7) T_3 and T_2 yield $(2, 2)$ to $(3, -2)$. The graph is pictured.

3. Let the triangle be ABC labeled so that angle A is the largest of the three angles and $|AB| \le |AC|$. Then angle A is at least $\pi/3$, so when it is bisected the two halves measure $\alpha \ge \pi/6$. Let the angle bisector at A be AD. By assumption, $|AD| > 1$. Now consider a new triangle $AB'C'$, where the angle bisector at A is AD as before, the two angles α are the same as before, but $B'C'$ is perpendicular to AD. If E is chosen on DC so that $|DE| = |DB|$, then $\text{Area}(\triangle DBB') = \text{Area}(\triangle DC'E) \le \text{Area}(\triangle DCC')$. Therefore $\text{Area}(\triangle ABC) \ge \text{Area}(\triangle AB'C')$. But $|DC'| = |AD| \tan\alpha > \tan\alpha$ and $|DB'| > \tan\alpha$, so $\text{Area}(\triangle AB'C') = (1/2)|AD||B'C'| > \tan\alpha \ge \tan(\pi/6) = 1/\sqrt{3}$ as claimed.



4. *See Problem 4 of the Gordon Exam.*

5. Yes. Writing the equation as $(y-z)(y+z) = 1999 - x^2$ shows us one way to find infinitely many solutions by taking x to be any even integer, $z = (1998 - x^2)/2$, and $y = z + 1$.

6. We may see (by induction) that $F_n > 0$ for all $n \ge 1$; therefore that $F_n = F_{n-1} + F_{n-2} > F_{n-1}$ for all $n \ge 3$; so it follows that $F_n \to \infty$ as $n \to \infty$. Also, $F_{n+1} = F_n + F_{n-1} < F_n + F_n = 2F_n$ for $n \ge 3$; that is, each Fibonacci number $\ge F_4 = 3$ is less than double the preceding Fibonacci number. Together with the fact that $F_n \to \infty$, this tells us that if $k \ge 3$ is any number, then there is a Fibonacci number (strictly) between $k/2$ and k. (There is a Fibonacci number $\ge k$, so there is a least Fibonacci number $\ge k$, and the preceding Fibonacci number is therefore $> k/2$ but $< k$.)

Now let us prove our main result by induction: We claim that any positive integer may be written as a sum of different Fibonacci numbers. First, $1 = F_1$, $2 = F_3$, and $3 = F_4$ are easy. Now assume that k is an integer ≥ 4 and that all integers $< k$ may be written as sums of different Fibonacci numbers. We must show that it follows that k also may be written as a sum of different Fibonacci numbers. Now by the above, there is a Fibonacci number F_n with $k/2 < F_n < k$. Thus $k - F_n$ is a positive integer smaller than k, so by our induction hypothesis it may be written as a sum of different Fibonacci numbers. Since $F_n > k/2$ we have $k - F_n < k/2$ and all of the Fibonacci numbers used in writing it are $< k/2$. So none of the Fibonacci numbers used in writing $k - F_n$ is F_n itself. Therefore, adding F_n we have written k as a sum of different Fibonacci numbers as required. This completes our proof by induction that any positive integer may be written as a sum of different Fibonacci numbers.

1. There are two zeros on the end. To see this, we compute the result mod 8 and mod 25. First, compute mod 8:

$1^{1999} \equiv 1$.

$2^3 = 8 \equiv 0$ so $2^{1999} \equiv 0$.

$3^2 = 9 \equiv 1$, so $3^{1999} = (3^2)^{999} \cdot 3 \equiv 3$.

$4^2 = 16 \equiv 0$ so $4^{1999} \equiv 0$.

Therefore $1^{1999} + 2^{1999} + 3^{1999} + 4^{1999} \equiv 1 + 0 + 3 + 0 = 4$. It is divisible by 4 but not by 8. Then compute mod 25:

$1^{1999} \equiv 1$.

$2^4 = 16$, $2^8 = 16^2 = 256 \equiv 6$, $2^9 \equiv 6 \cdot 2 = 12$, $2^{10} \equiv 12 \cdot 2 = 24 \equiv -1$, $2^{20} \equiv (-1)^2 = 1$.

So $2^{1999} = 2^{1980} \cdot 2^{10} \cdot 2^9 \equiv 1 \cdot (-1) \cdot 12 = -12$.

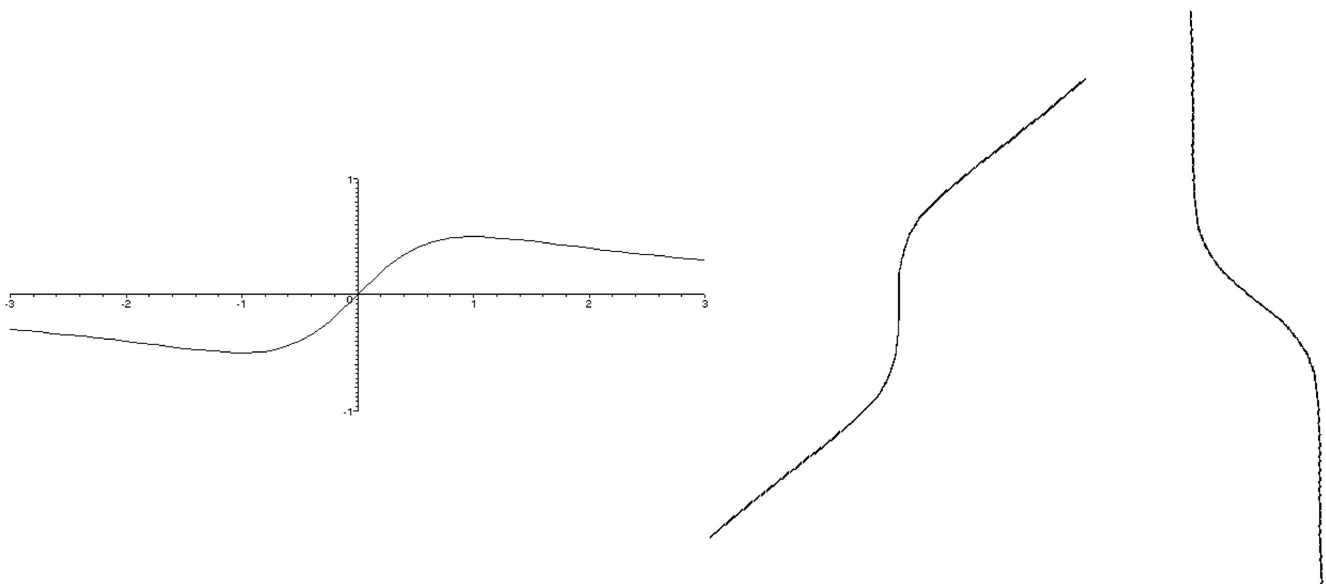
$3^3 = 27 \equiv 2$, $3^9 \equiv 2^3 = 8$, $3^{10} \equiv 8 \cdot 3 = 24 \equiv -1$, $3^{20} \equiv (-1)^2 = 1$. So $3^{1999} = 3^{1990} \cdot 3^{10} \cdot 3^9 \equiv 1 \cdot (-1) \cdot 8 = -8$.

$4^{1999} = (2^{1999})^2 \equiv (-12)^2 = 144 \equiv 19$.

Therefore $1^{1999} + 2^{1999} + 3^{1999} + 4^{1999} \equiv 1 - 12 - 8 + 19 = 0$. Since our number is divisible by 25 and by 4, it is divisible by their least common multiple 100. But since our number is not divisible by 8, it is not divisible by 1000. Thus its decimal expansion has exactly two zeros on the end.

2. The slope of the graph ranges from maximum value 1 (at the unique point $x = 0$) to minimum value $-1/8$ (at the two points $x = \sqrt{3}$ and $x = -\sqrt{3}$). So the angle a tangent to the curve makes with the horizontal ranges from maximum $\tan^{-1}(1) = \pi/4$ to minimum $\tan^{-1}(-1/8)$. We may rotate counterclockwise less than $\pi/4$ and the graph will still be the graph of a function. If we rotate exacly $\pi/4$, then the graph will have a vertical tangent line at one point, but will still be the graph of a function.

Similarly, for counterclockwise rotation the maximum value is $-\pi/2 + \tan^{-1}(1/8)$ (approximately -82.9 degrees). This rotated graph has two points with vertical tangent line.



3. Let us do this using complex numbers. Let the circle be the set of all complex numbers with $|z| = 1$, and let 1 be one of the vertices. Then the vertices are exactly the zeros of the polynomial $z^{17} - 1$. Call these zeros $w_0 = 1, w_1, w_2, \cdots, w_{16}$. The vertices other than 1 are then roots of the polynomial

$$p(z) = \frac{z^{17} - 1}{z - 1} = 1 + z + z^2 + \cdots + z^{15} + z^{16} = \prod_{k=1}^{16} (z - w_k).$$

Thus $L_1 \cdot L_2 \cdot \ldots \cdot L_{16} = |1 - w_1|\,|1 - w_2| \ldots |1 - w_{16}| = |p(1)| = |1 + 1 + \cdots + 1| = 17$.

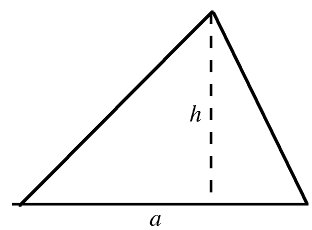
4. The important thing to notice is that $(6+\sqrt{37}\,)^{1999}+(6-\sqrt{37}\,)^{1999}$ is an integer. To see this, imagine expanding both powers by the binomial theorem. The terms involving odd powers of $\sqrt{37}$ occur with opposite signs and cancel. The terms involving even powers of $\sqrt{37}$ are all integers.

Now if N is that integer, then $(6+\sqrt{37}\,)^{1999}=N+(\sqrt{37}-6\,)^{1999}$ and $(\sqrt{37}-6\,)^{1999}>0$. We will be finished if we show that $(\sqrt{37}-6\,)^{1999}<10^{-1000}$. So it will be (more than) enough to show that $\sqrt{37}-6<1/10$. This is equivalent to $\sqrt{37}<6+1/10$, which is equivalent to $37<36+2\cdot 6\cdot(1/10)+1/100$, which is certainly true.

5.

$$\frac{2}{3!}+\frac{4}{5!}+\frac{6}{7!}+\cdots+\frac{2n}{(2n+1)!}+\cdots=\frac{1-1}{1!}+\frac{3-1}{3!}+\frac{5-1}{5!}+\frac{7-1}{7!}+\cdots+\frac{(2n+1)-1}{(2n+1)!}+\cdots$$
$$=\frac{1}{0!}-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}-\frac{1}{7!}+\cdots+\frac{1}{(2n)!}-\frac{1}{(2n+1)!}+\cdots=e^{-1}.$$

6. Consider first a triangle where all three sides have length ≤ 1. If one side has length a, what is the maximal possible length of the height h from that side? One of the two right triangles in the picture (the one with the longer portion of a as one leg) shows that h is at most $\sqrt{1-a^2/4}$.



Now consider a tetrahedron $ABCD$ where side AB is unknown, but the other five sides all have length ≤ 1. Then triangles ACD and BCD are both covered by the previous paragraph. Write a for the length of side CD. The volume of the tetrahedron is 1/3 times the area of triangle BCD times the height from face BCD. Now by the above, the area of triangle BCD is at most $(1/2)a\sqrt{1-a^2/4}$. The height of the tetrahedron is at most the height of triangle ACD from side CD (achieved only if ACD is perpendicular to BCD), and that is at most $\sqrt{1-a^2/4}$. So the volume of the tetrahedron is at most $(1/6)\cdot a\cdot(1-a^2/4)$. Now this function of a is increasing as a goes from 0 to 1 (for example by calculus), so the maximum volume occurs when $a=1$. This maximum is $(1/6)\cdot 1\cdot(1-1/4)=1/8$.