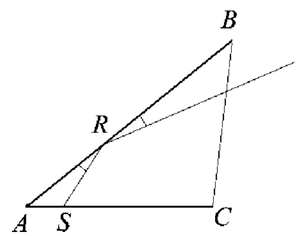
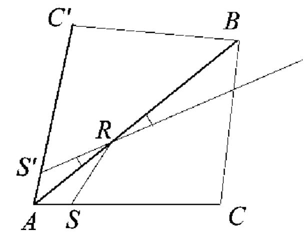
Rasor-Bareis Exam: solutions, page 1 February 12, 2000

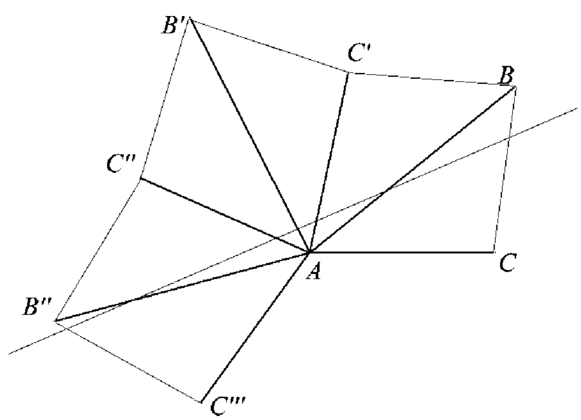
1. Put a Cartesian coordinate system in the plane so that the dodecagon is inscribed in the circle with center at the origin and radius 1, and point P_{11} is at $(1,0)$. Then $P_{12} = (\cos(\pi/6), \sin(\pi/6)) = (\sqrt{3}/2, 1/2)$, $P_1 = (\cos(\pi/3), \sin(\pi/3)) = (1/2, \sqrt{3}/2)$, $P_2 = (0,1)$, $P_4 = (\cos(5\pi/6), \sin(5\pi/6)) = (-\sqrt{3}/2, 1/2)$, $P_9 = (\cos(5\pi/3), \sin(5\pi/3)) = (1/2, -\sqrt{3}/2)$. With this choice of coordinates, the line P_1P_9 has equation $x = 1/2$ and the line P_4P_{12} has equation $y = 1/2$. So it only remains to show that the point $(1/2, 1/2)$ is on the line P_2P_{11}; but this is true since $(1/2, 1/2)$ is the midpoint of the segment P_2P_{11}.

2. Label the vertices so that after the ray crosses side BC into the triangle, the first side it hits is AB. Say it hits AB at point R. It is to reflect there. If the next side it reaches is side BC, then the ray leaves the angle and we are done. So assume the next hit S is on side AC. At the reflection point R, the angle of incidence should equal the angle of reflection, as indicated in the first figure.





Now reflect the plane through line AB. Triangle ABC is reflected to a triangle ABC' and the line segment RS is reflected to a segment RS'. Angle ARS equals angle ARS', so the angles marked in the second figure are equal, and this means that segment RS' is a continuation of the original ray in a straight line. Continue reflecting in sides through A: reflect ABC' in line AC' to get $AB'C'$; reflect $AB'C'$ in line AB' to get $AB'C''$, and so on. The reflected ray becomes a straight line l. Each time l crosses from one reflected triangle to another corresponds to a time the original ray is reflected within triangle ABC. Eventually l escapes, since it gets further from point A than the maximum of distances AB and AC. When l escapes, it does so by passing through one of the reflected images of side BC. But that means the original reflected ray escapes through the original side BC.



3. For any real number x, the three points $(x,1)$, $(x,2)$, and $(x,3)$ will give a triple of colors (RRR, RRB, RBR, RBB, BRR, BRB, BBR, or BBB). Since there are more than eight real numbers x, we can find two numbers x_1 and x_2 such that the corresponding triples of colors are the same. Now, the three points $(x_1,1)$, $(x_1,2)$, and $(x_1,3)$ cannot all have different colors, so two of them have the same color, say (x_1,y_1) and (x_1,y_2). Since x_1 and x_2 give the same color triple, (x_2,y_1) and (x_2,y_2) have the same color as (x_1,y_1) and (x_1,y_2); these four points give the desired rectangle.

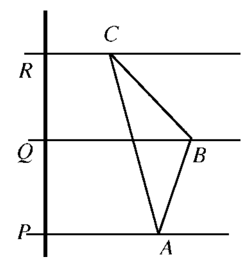
4. (a) Yes; an example is $p(x) = x + 60(x-1)(x-2) = 60x^2 - 179x + 120$.
(b) No. If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ where all of the coefficients are integers, then

$$p(3) - p(1) = a_n(3^n - 1^n) + a_{n-1}(3^{n-1} - 1^{n-1}) + \cdots + a_1(3^1 - 1^1).$$

Since all of the differences $3^k - 1^k$ are even, $p(3) - p(1)$ must be even. But $124 - 1$ is odd, so we cannot have $p(1) = 1$ and $p(3) = 124$.

5. Let $q_n = (\log 2)(\log 3)\ldots(\log n)/10^n$. Then $q_n/q_{n-1} = (\log n)/10$. So if $\log n < 10$, then q_n is smaller than q_{n-1}; if $\log n = 10$, then $q_n = q_{n-1}$; if $\log n > 10$, then q_n is greater than q_{n-1}. This means that the value of q_n gets smaller and smaller until n reaches 10^{10}, after which it remains the same for one step and grows thereafter. So q_n is the same for $n = 10^{10}$ and $n = 10^{10} + 1$, and this is the smallest value it takes.

6. Let the three points be A, B, C, where B is on the track between the other two. Choose a line l perpendicular to the racetracks far to one side, and let the points of l on the three racetracks be P, Q, R (in the same order as A, B, C). Then the area of the triangle ABC is the sum of the areas of two trapezoids $PQBA$ and $QRCB$ minus the area of trapezoid $PRCA$, or else the area of ABC is minus that number, depending on whether B is on one side or the other of AC. The area of trapezoid $PQBA$ is $PQ(PA+QB)/2$; but PQ is constant and PA, QB are both linear functions of time t, so this area is a linear function of t. Similarly the areas of the other two trapezoids are linear functions, so the area of triangle ABC has the form $f(t) = |\alpha t + \beta|$ for some constants α, β. (Note that the sign of $\alpha t + \beta$ may change if B happens to cross the line AC.) We know $f(0) = 2$ and $f(5) = 3$, so there are these possibilities:



$\beta = 2$,	$5\alpha + \beta = 3$,	so	$10\alpha + \beta = 4$;
$\beta = 2$,	$5\alpha + \beta = -3$,	so	$10\alpha + \beta = -8$;
$\beta = -2$,	$5\alpha + \beta = 3$,	so	$10\alpha + \beta = 8$;
$\beta = -2$,	$5\alpha + \beta = -3$,	so	$10\alpha + \beta = -4$.

So the final area is either 4 or 8.

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1. The proof is by induction on n. Begin with $n = 2$: either $1 \to 2$ or $2 \to 1$, so one of the two orders satisfies $a_1 \to a_2$. Now assume the result is true for $\{1, 2, \cdots, k\}$ and consider $k+1$. By hypothesis, there is some ordering $a_1 \to a_2 \to \cdots \to a_k$ for $\{1, 2, \cdots, k\}$. To add $k+1$, consdier two possibilities. If $a_i \to (k+1)$ for all i with $1 \le i \le k$, then add $a_{k+1} = k+1$ to get $a_k \to a_{k+1}$ as required. But if that is not true, let i_0 be the least i such that $a_i \to (k+1)$ fails. Then we have $a_1 \to \cdots \to a_{i_0-1} \to (k+1) \to a_{i_0} \to \cdots \to a_k$. So if we write: $b_i = a_i$ for $i < i_0$, $b_{i_0} = k+1$, and $b_i = a_{i-1}$ for $i > i_0$, we get $1, 2, \cdots, k+1$ listed in the order $b_1, \cdots, b_{k+1}$ and $b_1 \to b_2 \to \cdots \to b_{i_0-1} \to b_{i_0} \to b_{i_0+1} \to \cdots \to b_{k+1}$. Therefore, by induction, the result is true for any size n.

2. See Rasor-Bareis Exam, problem 2.

3. This sum is a number which, when written in base 6, has the form

$$0.1001000100001000001000000100000001000000001\ldots$$

(the 1's occur at positions $(n^2+3n-2)/2$ for $n = 1, 2, 3, \ldots$). The blocks of zeros increase in size, so this expansion is not eventually periodic, and therefore the number is irrational.

4. Let (α) stand for the rules of motion stated in the problem. Imagine a different set of rules (β), the same as (α) except that when two points meet, they simply pass each other, continuing in the original direction with the original speed. What is the difference between the outcomes of the two rules? Under the two rules, the positions, directions, and speeds of the heavy points are the same; the only difference is in which of the points are in which positions. Under rule (β), after a certain length of time T each heavy point has gone around the circle once and returned to its original position with its original direction. Therefore, under rule (α), after time T the heavy points are permuted among themselves. If we then continue to time $2T$, the same permutation of the points has been applied again. Since any permutation of a finite set has finite order (say N), we will arrive back to the starting state for all the points after time NT.

5. The proof is by mathematical induction on the degree of the polynomial. If $f(x) = c$ is an integer constant, then $\sum_{n=0}^{\infty} c/n! = ce$ as required. Now assume the result is true for all polynomials of degree d, and consider a polynomial $f(x)$ of degree $d+1$. If $f(0) = a$ is the constant term, then all other terms have a factor of x, so we may write $f(x) = xg(x) + a$, where $g(x)$ is a polynomial of degree d. If we substitute $x+1$ for x, we see that $h(x) = g(x+1)$ is also a polynomial of degree d. Now by the induction hypothesis, $\sum_{n=0}^{\infty} h(n)/n! = ke$ for some integer k, and then we have

$$\sum_{n=0}^{\infty} \frac{f(n)}{n!} = \sum_{n=0}^{\infty} \frac{nh(n-1)+a}{n!} = \sum_{n=1}^{\infty} \frac{nh(n-1)}{n!} + \sum_{n=0}^{\infty} \frac{a}{n!} = \sum_{n=1}^{\infty} \frac{h(n-1)}{(n-1)!} + ae = \sum_{n=0}^{\infty} \frac{h(n)}{n!} + ae = ke + ae = (k+a)e,$$

as required. So by induction the result is true for all integer polynomials.

6. Suppose z is a zero of $p(z)$ with $|z| \le 1$. Certainly $z \ne 1$, since $p(1) = \sum a_k > 0$ and $z \ne 0$ since $p(0) = a_0 > 0$. Multiplying out $0 = (1-z)p(z)$ and putting the constant term on one side by itself, we get

$$a_0 = (a_0 - a_1)z + (a_1 - a_2)z^2 + (a_2 - a_3)z^3 + \cdots + (a_{n-1} - a_n)z^n + a_n z^{n+1}.$$

Now recall that the complex absolute value satisfies: (i) $|ab| = |a|\,|b|$; and (ii) $|\sum u_i| \le \sum |u_i|$ with equality only if all nonzero u_i's are positive multiples of the others. So

$$\begin{aligned} a_0 &= |a_0| \\ &= \left|(a_0 - a_1)z + (a_1 - a_2)z^2 + (a_2 - a_3)z^3 + \cdots + (a_{n-1} - a_n)z^n + a_n z^{n+1}\right| \\ &\le (a_0 - a_1)\,|z| + (a_1 - a_2)\,|z|^2 + (a_2 - a_3)\,|z|^3 + \cdots + (a_{n-1} - a_n)\,|z|^n + a_n|z|^{n+1} \\ &\le (a_0 - a_1) + (a_1 - a_2) + (a_2 - a_3) + \cdots + (a_{n-1} - a_n) + a_n \\ &= a_0. \end{aligned}$$

Since the extremes are equal, both $\le$ are actually $=$. Since the first $\le$ is equality, and all $(a_k - a_{k+1})$ are positive, we conclude that z^2 is a positive multiple of z, so z is real and positive. Since the second $\le$ is equality, we conclude that $|z| = 1$. Therefore $z = 1$, which has already been ruled out. This contradiction shows that the assumption that there exists a zero z with $|z| \le 1$ is impossible.