An Brief Introduction to Billiards

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"What is?" Seminar - 7/6/09

What is billiards? It's the study of, on surfaces with boundary, geodesic paths subject to the classical reflection rule on the boundary: the angle of incidence equals the angle of reflection. In this talk, we'll only concern ourselves with simple regions of the plane bounded by piecewise smooth curves - so in between reflections, the path of the billiard is a straight line segement. Line segments are moderately well understood by now, hence we can understand a billiard path as well as we can understand its reflections.

One effective way to deal with this reflection process is to set it up as a dynamical system. Many major results and open problems in the theory of billiards are best expressed in this language.

Definition: The *phase space* P of a billiard region S is the cylinder $(\mathbb{R}/s\mathbb{Z}) \times [-1, 1]$, where s is the arclength of ∂S. A point $(\varphi, x) \in P$ corresponds to a reflection at arclength index φ with exit angle $\theta = \arccos x$ (i.e., x is the cosine of the exit angle).

Definition: The *billiard map* of a billiard region S is the map $B : P \to P$ defined by taking a reflection to the next reflection. That is, $(\varphi, x) \in P$ defines an outgoing ray that must hit the boundary again at some point φ' with an exit angle whose cosine is x'. Then $B(\varphi, x) = (\varphi', x')$.

A Brief Word on Periodic Orbits

Definition: A *Birkhoff billiard* is a smooth (C^∞), convex billiard region.

Birkhoff billiards have the nice properties that the billiard map B is smooth and area-preserving on P.

Definition: The *orbit* of (φ, x) is the sequence $(B^n(\varphi, x))_{n=0}^{\infty}$. An orbit is *k-periodic* if $\forall n \in \mathbb{N}$, $B^{n+k}(\varphi, x) = B^n(\varphi, x)$, and k is minimal among all such numbers; an orbit is *periodic* if it is periodic for some k. The rotation number of a period is, vaguely, the (positive) number of times around the circle the path passes in one period.

Lemma: The k-periodic orbits correspond to the critical points of the length functional $L(p_{1,\dots,}p_k) = \sum_{i=1}^{k} d(p_i, p_{i-1})$, $p_i \in \partial S$.

Theorem: (Birkhoff) In a Birkhoff billiard, $\forall k > 1, \forall l : 1 \leq l \leq \frac{k}{2}$ coprime to k, there are at least two k-periodic orbits with rotation number l.

Open Question: Is, for all k, the set of k-periodic orbits nowhere dense (in the phase space)?

For many other types of billiard regions, the question of existence of periodic orbits is still open. It is still unknown whether every convex polygonal billiard has a periodic orbit at all - even for triangles!

A Brief Word on "Nearby Paths"

The rate at which nearby paths diverge can be measured by considering $|\det dB^n|$ as n grows large.

Definition: The *Lyapunov exponent* at a point in the phase space is $\lambda(\varphi, x) = \limsup_{n\to\infty} \frac{1}{n}\log(|\det dB^n|)$. If $\lambda = 0$ at a point (φ, x), then nearby orbits diverge from that of (φ, x) at a reasonable (sub-exponential) rate. If $\lambda = c > 0$, nearby orbits tend to spread away at an exponential rate ($\approx e^{\lambda n}$).

Definition: A billiard is *chaotic* if the set $\{(\varphi, x) \in P : \lambda(\varphi, x) > 0\}$ has positive measure.

Theorem: Billiards in Bunimovich's stadium are chaotic.

Compare to the circle and the ellipse, in which paths diverge no faster than linearly.

Open Question: Is there a chaotic Birkhoff billiard?

Further Reading

Katok, A.B. "Billiard Table as Mathematician's Playground."
available via google
Koslov, V., Treshchëv, D. *Billiards: A Genetic Introduction.*
available through the OSU library
Tabachnikov, S. *Billiards.*
available through the OSU library

For those expecting a talk on Alhazen's problem, the second reference below gives one of the nicer solutions to the problem that I encountered. The first reference below gives a proof of the interesting fact that almost every solution to Alhazen's problem is not constructible via compass and edge.

Neumann, P. "Reflections on Reflection in a Spherical Mirror." Am. Math. Monthly 105:6 (1998).
accessible by JSTOR
Waldvogel, J. "The Problem of the Circular Billiard." Elemente der Mathematik 47 (1992).
accessible by the OSU online journals list