BOHR COMPACTIFICATIONS

Synopsis

In the early 20th century, it was discovered that much of the classical theory of Fourier analysis on $\mathbb{R}$ could be generalized to functions with domains in other additive groups similar to $\mathbb{R}$. The properties necessary for a group to be similar enough to $\mathbb{R}$ to produce an appropriate generalization were eventually worked out: namely, such a group must be abelian and have a topology which is locally compact and Hausdorff. The fruits of this program were significant and form some of the basis for the field of harmonic analysis.

The Bohr compactification of such a group is a tool generally used in the study of almost periodic functions on that group, which are generalizations of almost periodic functions on $\mathbb{R}$. These notes will deal with a small section of knowledge required to define and prove some basic results about the Bohr compactification. Hopefully this will provide some of the flavor of the generalization program.

The only prior knowledge assumed is that of basic general topology and group theory. Some theorems are stated without proof; this is not to mean they are inaccessible, only that their proofs are too long or require too much background work to be included. Some possible references are provided at the end. Obviously none of the results are my own, and the original discoverers deserve due credit (although no historical notes are included).

Locally compact Hausdorff abelian groups

The space $\mathbb{R}$ is gifted with a a large number of different structures which influence each other in interesting ways. Here we isolate two properties essential to our program: $\mathbb{R}$ is an abelian group under addition, and it has a non-trivial topology which interacts nicely with the group structure.

Definition. A *topological group* is a group G endowed with a topology with respect to which the group operations, $\cdot : G \times G \to G$ and ${}^{-1} : G \to G$, are continuous. A *locally compact Hausdorff abelian group* (or *LCA group*) is a topological group which is abelian as a group and both locally compact and Hausdorff as a topological space.

You can check that both $\mathbb{R}$ and $\mathbb{C}$ are LCA groups under addition given their standard metric topologies. Another important example of an LCA group is the circle group $\mathbb{T}$ of complex numbers with norm 1 taken as a group under multiplication, with the relative topology inherited from $\mathbb{C}$. The circle group is the basis for a useful tool:

Definition. A *character* on a group G is a homomorphism from G to the circle group, $\mathbb{T}$.

The set of all characters on G is itself a group under pointwise multiplication; i.e. for two characters γ and δ, and for $g \in G$, we let $(\gamma\delta)(g) = \gamma(g)\delta(g)$ and $\gamma^{-1}(g) = \gamma(g)^{-1}$. Note that since $\mathbb{T}$ has an appropriate topology, if G is a topological group it makes sense to discuss the *continuous* characters on G. You can verify that the continuous characters on a topological group G also form a group; this group plays an important role and is given a special name:

Definition. The *dual group* of a topological group G is the group of all continuous characters on G and is denoted $\widehat{G}$.

We restrict ourselves to mainly discussing continuous characters for the important reason that $\widehat{G}$ can be topologized in a nice way relating to the topology of G. For any open subset $V \subseteq \mathbb{T}$ and any compact subset $K \subseteq G$, define

$$U(K, V) = \left\{ \gamma \in \widehat{G} : \gamma(K) \subseteq V \right\}$$

and let this act as a sub-basis for a topology on $\widehat{G}$. This topology is known as the *compact-open* topology on $\widehat{G}$ for obvious reasons. It turns out that if G is an LCA group, this topology also makes $\widehat{G}$ an LCA group; the same would generally not be true if we applied this method to the group of all characters on G. Proofs of these facts as well as those of other unproved statements in this section can be found in the references.

We now need some more knowledge about the nature of character groups.

Theorem. *Let G be an LCA group. If G is compact, then $\widehat{G}$ is discrete in the compact-open topology. Similarly, if G is discrete, then $\widehat{G}$ is compact.*

As an example of the above principle, observe that the dual of $\mathbb{Z}$ with the discrete topology is simply $\mathbb{T}$, since every character on $\mathbb{Z}$ is defined by where it sends 1. You can check the compact-open topology induced on $\mathbb{T}$ is the same as the relative topology inherited from $\mathbb{C}$, so that $\mathbb{T}$ is compact in this topology as desired.

It is a more difficult exercise to find $\widehat{\mathbb{T}}$, but it turns out that this is again $\mathbb{Z}$ in the discrete topology. There is a very interesting structure theorem which clarifies this phenomenon:

Theorem (Pontryagin Duality). *Let G be an LCA group. The map $\tau : G \to \widehat{\widehat{G}}$ such that $\tau(g)(\gamma) = \gamma(g)$ for $g \in G$, $\gamma \in \widehat{G}$ is simultaneously an isomorphism of groups and a homeomorphism.*

Such maps which are simultaneously isomorphisms and homeomorphisms are sometimes given a special name:

Definition. For topological groups G and H, a *topological isomorphism* is a function $\rho : G \to H$ which is simultaneously a homeomorphism and a group isomorphism.

Bonus Exercise. Let G and H be topological groups which are both homeomorphic and isomorphic, although possibly through two different functions. Are G and H necessarily topologically isomorphic?

We will also need in the next section the following lemma used in the proof of the Pontryagin duality theorem:

Lemma. *Let G be an LCA group. $\forall g_1, g_2 \in G$ with $g_1 \neq g_2$, there is a character $\gamma \in \widehat{G}$ such that $\gamma(g_1) \neq \gamma(g_2)$. As a consequence, if G is non-trivial then $\widehat{G}$ is non-trivial.*

We often express this property by saying the set of continuous characters *separates points.*

The Bohr compactification of an LCA group

We now define the Bohr compactification of an LCA group G in such a way as might come from a clever observation; if G were compact, we know from the theorems of the last section that $\widehat{\widehat{G}} \cong G$ is also compact. In some sense, $\widehat{\widehat{G}}$ is compact *because* $\widehat{G}$ is discrete. In general, if we take $\widehat{G}$ as a group arbitrarily assigned with the discrete topology (denoted $\widehat{G}_d$), then its dual will be compact.

Definition. The *Bohr compactification* of an LCA group G, denoted bG, is the dual of $\widehat{G}_d$.

This definition is not revealing; we have no idea in what sense bG might be a "'compactification"' of G. Pontryagin duality shows that G is isomorphic to the group of continuous characters on $\widehat{G}$, i.e. $\widehat{\widehat{G}}$, while bG is by definition the group of all characters (since any function is continuous when its domain is discrete). Thus G maps injectively into bG in a canonical way, but how the topologies on the two groups relate is initially unclear. The following theorem illustrates the nature of this map:

Main Theorem 1. *Let G be an LCA group and let $\sigma : G \to bG$ be the natural map from G into bG defined by $\sigma(g)(\gamma) = \gamma(g)$ for $g \in G$, $\gamma \in \widehat{G}$. Then σ is an injective continuous homomorphism of groups, and $\sigma(G)$ is dense in bG.*

Proof. Step 1. First we show that σ is an isomorphism of groups. Remember that bG is the group of all characters on $\widehat{G}$. We can easily see σ is a homomorphism; if $\gamma \in \widehat{G}$, $g, h \in G$, then

$$\sigma(gh)(\gamma) = \gamma(gh) = \gamma(g)\gamma(h) = \sigma(g)(\gamma)\sigma(h)(\gamma)$$

or in other words, $\sigma(gh) = \sigma(g)\sigma(h)$. Additionally, σ can be viewed as the composition of the isomorphism $\tau : G \to \widehat{\widehat{G}}$ and the inclusion $\iota : \widehat{\widehat{G}} \hookrightarrow bG$ which are both injective. Thus σ is also injective, and therefore $G \cong \sigma(G)$.

Step 2. Next we show that $\sigma(G)$ is dense in bG. Assume not, then $\overline{\sigma(G)}$ is a proper subset of bG, and is in fact a subgroup(the product of limits of sequences is just the limit of the products by continuity). Since bG is abelian, we may examine $bG/\overline{\sigma(G)}$. You can verify that this group is compact Hausdorff under the quotient topology; thus since it is non-trivial there must be a non-trivial character δ on the group. Therefore $\tilde{\delta}(\alpha) = \delta(\alpha + \overline{\sigma(G)})$ for $\alpha \in bG$ defines a non-trivial character on bG which is 1 on $\overline{\sigma(G)}$. By Pontryagin duality, this character corresponds to an element $\gamma \in \widehat{G}_d$ which is non-trivial. If τ is the Pontryagin dual mapping from $\widehat{G}_d$ to $\widehat{bG}$ then by definition, $\forall g \in G$

$$\gamma(g) = \sigma(g)(\gamma) = \tau(\gamma)(\sigma(g)) = 1$$

since $\tau(\gamma) = \tilde{\delta}$ is 1 on $\overline{\sigma(G)}$. But this is a contradiction since γ is non-trivial; thus we must have had $\sigma(G)$ dense in bG.

Step 3. Finally we show that σ is continuous. Note that the compact sets in $\widehat{G}_d$ are just the finite sets, since any infinite set has an open covering by singleton sets. Thus, using the definition of the compact-open topology, there is a sub-basis on bG composed of sets of the form

$$U_{bG}(\{\gamma_1, \ldots, \gamma_n\}, V) = \{\alpha \in bG : \{\alpha(\gamma_1), \ldots, \alpha(\gamma_n)\} \subseteq V\}$$

where $V \subseteq \mathbb{T}$ is open and $\gamma_1, \ldots, \gamma_n \in \widehat{G}$. If we decompose σ as $\iota \circ \tau$, then it suffices to show that ι is continuous since τ is a homeomorphism. But the set

$$U_{\widehat{\widehat{G}}}(\{\gamma_1, \ldots, \gamma_n\}, V) = \left\{\beta \in \widehat{\widehat{G}} : \{\beta(\gamma_1), \ldots, \beta(\gamma_n)\} \subseteq V\right\}$$

is open in $\widehat{\widehat{G}}$ since finite subsets are compact in any topological space. Clearly $\iota^{-1}(U_{bG}) = U_{\widehat{\widehat{G}}}$. Thus the pre-images of all sub-basis elements under ι are open, and therefore the pre-images of all open sets are open. So ι is continuous as desired. $\square$

ALMOST PERIODIC FUNCTIONS

As might be expected, the almost periodic functions on $\mathbb{R}$ are functions which in some sense "'almost"' have periods.

Definition. An *almost periodic* function $f : \mathbb{R} \to \mathbb{C}$ is one which has the property that $\forall \epsilon > 0$, there is an l_ϵ such that for any interval I of length l_ϵ there exists a $t \in I$ such that

$$\sup_{x \in \mathbb{R}} \|f(x) - f(x+t)\| < \epsilon$$

This definition is meant to mirror the fact that for a periodic function, all intervals of a certain size contain some period. These functions are sometimes known as *uniformly almost periodic* or *Bohr almost periodic* to distinguish them from other similarly named functions. It is a theorem that the almost periodic functions are exactly the closure under the uniform norm of the set of functions of the form

$$f(x) = \sum_{n=-m}^{m} a_n e^{i\lambda_n x}$$

where $a_n \in \mathbb{C}, \lambda_n \in \mathbb{R}$. Observing that the functions $e^{i\lambda_n x}$ are characters of $\mathbb{R}$, we take this theorem as inspiration for our generalized definition:

Definition. The set of *almost periodic* functions on an LCA group G, denoted $AP(G)$, is the closure in the uniform topology on $C(G, \mathbb{C})$ of the set of linear combinations of continuous characters on G. Here $C(G, \mathbb{C})$ indicates the set of continuous functions from G to $\mathbb{C}$.

As with $\mathbb{R}$, there are other equivalent definitions for which our definition is a theorem; further information can be found in the references. Notice that the topology used here, the uniform topology, is in general not equivalent to the compact-open topology of the previous section; it is sometimes coarser. The main theorem relating $AP(G)$ to bG is as follows:

Main Theorem 2. *Let G be an LCA group, and let σ be the natural injection of G into bG. A function $f : G \to \mathbb{C}$ is almost periodic if and only if there is a continuous function $h : bG \to \mathbb{C}$ such that $f = h|_{\sigma(G)} \circ \sigma$.*

Proof. Step 1. First we prove that every continuous character on G extends to a continuous character on bG (viewing G as the subset $\sigma(G)$). Let $\gamma \in \widehat{G}$; γ induces a continuous homomorphism $\gamma^* : \widehat{\mathbb{T}} \to \widehat{G}$ by $\gamma^*(\rho) = \rho \circ \gamma$ where $\rho \in \widehat{\mathbb{T}}$. We know that $\mathbb{T}$ is compact, thus $\widehat{\mathbb{T}}$ is discrete and so γ^* is also continuous when interpreted as a homomorphism onto

$\widehat{G}_d$. Using the same method again, we get a continuous homomorphism $\gamma^{**} : bG \to \widehat{\widehat{\mathbb{T}}}$. Let $\tau : \mathbb{T} \to \widehat{\widehat{\mathbb{T}}}$ be the Pontryagin duality mapping. If $g \in G$, $\rho \in \widehat{\mathbb{T}}$ then

$$\begin{aligned}\gamma^{**}(\sigma(g))(\rho) &= (\sigma(g) \circ \gamma^*)(\rho) = \sigma(g)(\gamma^*(\rho)) \\ &= \sigma(g)(\rho \circ \gamma) = (\rho \circ \gamma)(g) \\ &= \rho(\gamma(g)) = \tau(\gamma(g))(\rho)\end{aligned}$$

Thus $\tau(\gamma(g)) = \gamma^{**}(\sigma(g))$, and so $\tau^{-1} \circ \gamma^{**}$ is the desired character.

Step 2. If f is almost periodic, it is a uniform limit of linear combinations of continuous characters. Each character involved can be extended to a continuous character on bG by the above; the terms in the limit can therefore be extended by taking the extension of each component character. The uniform limit of these extensions exists since they converge uniformly on the dense subset $\sigma(G)$; call it h. h is continuous on bG, and its restriction to $\sigma(G)$ is f by definition. Thus $f = h|_{\sigma(g)} \circ \sigma$ as desired.

Step 3. To prove the reverse inclusion, we will need a version of the Stone-Weierstrass theorem:

Theorem (Stone-Weierstrass). *Let X be a compact Hausdorff space and let A be a set of continuous functions from X to $\mathbb{C}$ closed under the operations of addition, multiplication by scalars and other functions in A, and complex conjugation. Also, let A have the additional property that $\forall x_1, x_2 \in X$ with $x_1 \neq x_2$, there exists a function $\alpha \in A$ such that $\alpha(x_1) \neq \alpha(x_2)$. Then A is dense in $C(X, \mathbb{C})$ under the uniform topology.*

A proof of this theorem can be found in standard analysis texts.

We apply this theorem to the set of all linear combinations of continuous characters on bG. It is straightforward to verify that linear combinations of characters satisfy the desired closure properties. Additionally, by an earlier theorem we know that continuous characters separate points, and thus so do their linear combinations. Therefore, the Stone-Weierstrass theorem applies and the set of all linear combinations of continuous characters is dense in $C(bG, \mathbb{C})$ under the uniform topology. In other words, every continuous complex-valued function h on bG is a uniform limit of linear combinations of characters. If we restrict a character on bG to $\sigma(G)$, it is a character on $\sigma(G)$; thus $h|_{\sigma(G)}$ can be expressed as a uniform limit of linear combinations of continuous characters on $\sigma(G)$. Therefore $h|_{\sigma(G)} \circ \sigma$ is also of this form since σ is a continuous isomorphism, and thus is almost periodic by definition. $\square$

Note that in the course of proving this theorem, we also showed that all continuous functions on bG are almost periodic; thus there is nothing more to be gained by discussing almost periodic functions on bG. In particular, the almost periodic functions on any compact LCA group are just the continuous functions, so the idea is only interesting on non-compact LCA groups.

The supposed usefulness of this theorem is that it allows us to talk about almost periodic functions in terms of continuous functions on a compact space, which we understand very well. If we have a concrete form for the continuous complex-valued functions on bG then we automatically have a wealth of information about almost periodic functions on G. Even if this is not the case, we may be able to prove things about $C(bG, \mathbb{C})$ and then pull those things back to $AP(G)$.

We end with another characterization of almost periodic functions which we will not prove.

Theorem. *If G is an LCA group, an equivalent criterion for $f : G \to \mathbb{C}$ to be almost periodic is that f be continuous and bounded, and the closure of $\{f_g(x) = f(gx) \in C(G, \mathbb{C}) : g \in G\}$ under the uniform norm be compact.*

This is often taken as a definition of almost periodicity, as it more clearly represents our feelings about such functions; as they are shifted, they do not vary too wildly and are thus constrained to lie in a compact set. This definition might lead to questions about the dynamical nature of $AP(G)$.

Possible Reading

· *A Course in Abstract Harmonic Analysis* by G.B. Folland
· *Abstract Harmonic Analysis* by Hewitt and Ross
· *Fourier Analysis on Groups* by Walter Rudin
· *Pontryagin Duality & the Structure of Locally Compact Abelian Groups* by Morris