What is?

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An elementary proof that the image of a non-atomic finite measure is a closed interval

Let (S, Σ, μ) be a measure space. A set $A \in \Sigma$ is called an atom if $\mu(A) > 0$ and for each $B \in \Sigma$ with $B \subset A$ either $\mu(B) = 0$ or $\mu(B) = \mu(A)$. Note that if A is not an atom, then there is some $B, C \in \Sigma$ such that $B, C \subset A$ and $\mu(B) \in (0, \frac{1}{2}\mu(A)]$ and $\mu(C) \in [\frac{1}{2}\mu(A), \mu(A))$. A non-atomic measure is defined as a measure without atoms. Define $\mu(\Sigma)$ as $\{\mu(A) : A \in \Sigma\}$.

Theorem. *If μ is a non-atomic finite measure, then $\mu(\Sigma) = [0, \mu(S)]$.*

Proof. For simplicity, we may assume that $\mu(S) = 1$, or else we could look at the non-atomic finite measure $\frac{1}{\mu(S)}\mu$.

Subclaim 1. *There is some $A \in \Sigma$ such that $\mu(A) \in (\frac{1}{4}, \frac{1}{2}]$.*

Proof of Subclaim. We will show the stronger result that for any $B \in \Sigma$ with $\mu(B) > 0$, there is some $A \in \Sigma$ with $A \subset B$ and $\mu(A) \in (\frac{1}{4}\mu(B), \frac{1}{2}\mu(B)]$. Let $B \in \Sigma$ with $\mu(B) > 0$. Assume that there is not such an $A \in \Sigma$ with $A \subset B$ and $\mu(A) \in (\frac{1}{4}\mu(B), \frac{1}{2}\mu(B)]$. If there was an $A \in \Sigma$ with $A \subset B$ and $\mu(A) \in [\frac{1}{2}\mu(B), \frac{3}{4}\mu(B))$, then $B \backslash A \subset B$ and $\mu(B \backslash A) \in (\frac{1}{4}\mu(B), \frac{1}{2}\mu(B)]$. So for each $A \subset B$ either $\mu(A) \in [0, \frac{1}{4}\mu(B)]$ or $\mu(A) \in [\frac{3}{4}\mu(B), \mu(B)]$.

The set $\mathcal{C} = \{A \in \Sigma : A \subset B \text{ and } \mu(A) \geq \frac{3}{4}\mu(B)\}$ is not empty, since $B \in \mathcal{C}$. Also, $\mathcal{C}$ is closed under finite intersection, since if $C, D \in \mathcal{C}$, then $\mu(C \cap D) = \mu(C) + \mu(D) - \mu(C \cup D) \geq (\frac{3}{4} + \frac{3}{4} - 1)\mu(B) = \frac{1}{2}\mu(B)$. By assumption, if $\mu(C \cap D) \geq \frac{1}{2}\mu(B)$, then $\mu(C \cap D) \geq \frac{3}{4}\mu(B)$, and so $C \cap D \in \mathcal{C}$.

Define α by

$$\alpha = \inf_{A \in \mathcal{C}} \mu(A)$$

So $\alpha \geq \frac{3}{4}\mu(B)$ and for each $n \in \mathbb{N}$, there is some $A_n \in \mathcal{C}$ such that $\mu(A_n) < \alpha + \frac{1}{n}$. Define B_n recursively by $B_1 = A_1$ and $B_{n+1} = B_n \cap A_{n+1}$. Since $\mathcal{C}$ is closed under finite unions, B_n is a decreasing sequence of sets in $\mathcal{C}$, and for each $n \in \mathbb{N}$, $\alpha \leq \mu(B_n) < \alpha + \frac{1}{n}$.

Let $G = \bigcap_n B_n$. So $\alpha \leq \mu(G) < \alpha + \frac{1}{n}$ for each n. Hence $\mu(G) = \alpha$. So $G \in \mathcal{C}$, and since G is not an atom, there is some $H \subset G$ such that $\mu(H)$ is in $[\frac{\alpha}{2}, \alpha)$. Since $\alpha \geq \frac{3}{4}\mu(B)$, $\frac{\alpha}{2} \geq \frac{3}{8}\mu(B)$ and so by assumption, $H \in \mathcal{C}$ contradicting minimality of α.

Subclaim 2. *For any $B \in \Sigma$ with $\mu(B) \in (\frac{1}{4}, \frac{1}{2}]$ and for any $n \in \mathbb{N}$, there is some $C \in \Sigma$ such that $C \subset S \backslash B$ and $\mu(C) \in [\frac{2}{4^n}, \frac{3}{2 \cdot 2^n})$*

Proof of Subclaim. Use induction on n. So $\mu(S \backslash B) \in [\frac{1}{2}, \frac{3}{4})$. For $n = 1$, we may choose $C = S \backslash B$. Let $n \geq 1$ and suppose there is some $D \subset S \backslash B$ where $\mu(D) \in [\frac{2}{4^n}, \frac{3}{2 \cdot 2^n})$. By applying the stronger statement in the proof of the previous subclaim, there is some $C \subset D$ where $\mu(C) \in (\frac{1}{4}\mu(D), \frac{1}{2}\mu(D)]$, but since $\mu(D) \in [\frac{2}{4^n}, \frac{3}{2 \cdot 2^n})$, $\mu(C) \in [\frac{2}{4^{n+1}}, \frac{3}{2 \cdot 2^{n+1}})$. Since $C \subset D$, $C \subset S \backslash B$.

Subclaim 3. *For each $C \in \Sigma$ such that $\frac{1}{4} < \mu(C) < \frac{1}{2}$ and for each $n \in \mathbb{N}$, there is some $D \in \Sigma$ such that $C \subset D$ and $\frac{1}{2} - \frac{3}{2 \cdot 2^n} \leq \mu(D) < \frac{1}{2}$.*

Proof of Subclaim. Suppose $C \in \Sigma$ such that $\frac{1}{4} < \mu(C) < \frac{1}{2}$ and $n \in \mathbb{N}$. We will define an increasing finite sequence of sets D_m, recursively, where $m \in \{0, 1, 2, \ldots, 4^{n-1}\}$. Let $D_0 = C$. Suppose D_k for $k \in \{0, 1, \ldots, m\}$ is defined and is an increasing sequence of sets. If $\mu(D_m) + \frac{3}{2 \cdot 2^n} \geq \frac{1}{2}$, then let

$D_{m+1} = D_m$. Else $\mu(D_m) + \frac{3}{2\cdot 2^n} < \frac{1}{2}$ and so $\frac{1}{4} < \mu(D_m) < \frac{1}{2}$. We may apply the second subclaim. So there is some $E_m \subset S \backslash D_m$ such that $\mu(E_m) \in [\frac{2}{4^n}, \frac{3}{2\cdot 2^n})$. Define $D_{m+1} = D_m \cup E_m$.

Hence $\mu(D_m) + \frac{2}{4^n} \leq \mu(D_{m+1}) < \mu(D_m) + \frac{3}{2\cdot 2^n} < \frac{1}{2}$.

So from either case for the definition, $D_m \subset D_{m+1}$ and since $\mu(C) < \frac{1}{2}$, each $\mu(D_m) < \frac{1}{2}$

Suppose one of the D_m satisfies $\mu(D_m) + \frac{3}{2\cdot 2^n} \geq \frac{1}{2}$, then let k be the least such m. So $C \subset D_k$ and $\mu(D_k) + \frac{3}{2\cdot 2^n} \geq \frac{1}{2}$ yields $\frac{1}{2} - \frac{3}{2\cdot 2^n} \leq \mu(D_k)$ and $\mu(D_k) < \frac{1}{2}$. We can choose $D = D_k$.

Suppose none of the D_m satisfies $\mu(D_m) + \frac{3}{2\cdot 2^n} \geq \frac{1}{2}$, then for each $m \in \{0, 1, 2, \ldots, 4^{n-1} - 1\}$, $\mu(D_m) + \frac{2}{4^n} \leq \mu(D_{m+1})$. For each $m \in \{0, 1, 2, \ldots, 4^{n-1}\}$, $\mu(C) + \frac{2}{4^n} \cdot m \leq \mu(D_m)$. For $m = 4^{n-1}$, $\mu(C) + \frac{1}{2} \leq \mu(D_m)$, and so $\mu(D_m) \geq \frac{1}{2}$ which is a contradiction. Hence one of the D_m must satisfy $\mu(D_m) + \frac{3}{2\cdot 2^n} \geq \frac{1}{2}$.

Subclaim 4. *There is some $E \in \Sigma$ such that $\mu(E) = \frac{1}{2}$.*

Proof of Subclaim. By the first subclaim, there is some $A \in \Sigma$ such that $\mu(A) \in (\frac{1}{4}, \frac{1}{2}]$. If $\mu(A) = \frac{1}{2}$, choose $E = A$. If $\mu(A) < \frac{1}{2}$, then we define a sequence of increasing sets B_n by recursion. Let $B_0 = A$. Then by the third subclaim, there is some B_1 such that $B_0 \subset B_1$ and $\frac{1}{2} - \frac{3}{2\cdot 2^1} \leq \mu(B_1) < \frac{1}{2}$. and so $\mu(B_1) \in (\frac{1}{4}, \frac{1}{2})$. We apply the third subclaim again, to obtain some B_2 such that $B_1 \subset B_2$ and $\frac{1}{2} - \frac{3}{2\cdot 2^2} \leq \mu(B_2) < \frac{1}{2}$ and moreover $\mu(B_2) \in (\frac{1}{4}, \frac{1}{2})$. And so on.

So B_n is an increasing sequence of sets such that for each $m \in \mathbb{N}$, $\frac{1}{2} - \frac{3}{2\cdot 2^m} \leq \mu(B_m) < \frac{1}{2}$.

Let $E = \bigcup_n B_n$. Then for each $n \in \mathbb{N}$, $\frac{1}{2} - \frac{3}{2\cdot 2^n} \leq \mu(E) \leq \frac{1}{2}$. Hence $\mu(E) = \frac{1}{2}$.

Denote the dyadic rationals in $[0, 1]$ by $\mathbb{Q}_2 := \{\frac{m}{2^n} : m \in \mathbb{Z}, n \in \mathbb{N}\} \cap [0, 1]$.

Subclaim 5. *There is a collection of subsets, $(A_r)_{r\in\mathbb{Q}_2}$, of S indexed by the dyadic rationals such that $\mu(A_r) = r$ and if $r < s$, then $A_r \subset A_s$.*

Proof of Subclaim. We will define the sequence of sets indexed by recursion. In the n^{th} step, we will define $A_{\frac{m}{2^n}}$ for each $m \in \{1, 3, , 5 \ldots, 2^n - 1\}$ where for $k, m \in \{0, 1, \ldots, 2^n\}$ with $k \leq m$, $A_{\frac{k}{2^n}} \subset A_{\frac{m}{2^n}}$ and $\mu(A_{\frac{k}{2^n}}) = \frac{k}{2^n}$

Define $A_0 = \emptyset$ and $A_1 = S$. So $A_0 \subset A_1$.

Suppose we have define such sets $A_{\frac{m}{2^n}}$ for each $m \in \{0, 1, \ldots, 2^n\}$. We wish to define $A_{\frac{2k+1}{2^{n+1}}}$ for each $k \in \{0, 1, \ldots, 2^n - 1\}$.

Let $k \in \{0, 1, \ldots, 2^n - 1\}$ and let $\mathrm{C} = A_{\frac{k+1}{2^n}} \backslash A_{\frac{k}{2^n}}$.

The measure $\nu = 2^n \cdot \mu \mid_C$ is a non-atomic measure with $\nu(C) = 1$. So there is some $E \subset C$ such that $\nu(E) = \frac{1}{2}$ by the fourth subclaim. Let $A_{\frac{2m+1}{2^{n+1}}} = A_{\frac{m}{2^n}} \cup E$. Then $\mu(A_{\frac{2m+1}{2^{n+1}}}) = \frac{m}{2^n} + \frac{1}{2^n} \cdot \frac{1}{2} = \frac{2m+1}{2^{n+1}}$ and $A_{\frac{m}{2^n}} \subset A_{\frac{2m+1}{2^{n+1}}} \subset A_{\frac{m+1}{2^n}}$.

Now we can finish the proof to the theorem.

We have defined for each $\beta \in \mathbb{Q}_2$, a set A_β such that $\mu(A_\beta) = \beta$ and for each $\beta, \gamma \in \mathbb{Q}_2$ with $\beta < \gamma$, $A_\beta \subset A_\gamma$.

For each $c \in [0, 1]$, define a set $B_c = \bigcup_{\beta\leq c} A_\beta$. (Note: This union is countable and so $B_c \in \Sigma$). Also for any $\beta, \gamma \in \mathbb{Q}_2$ with $\beta \leq c \leq \gamma$, $A_\beta \subset B_c \subset A_\gamma$ and so $\beta \leq \mu(B_c) \leq \gamma$.

Hence $\mu(B_c) = c$.