What is?
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AN ELEMENTARY PROOF THAT THE IMAGE OF A NON-ATOMIC FINITE MEASURE IS A CLOSED INTERVAL

Let \((S, \Sigma, \mu)\) be a measure space. A set \(A \in \Sigma\) is called an atom if \(\mu(A) > 0\) and for each \(B \in \Sigma\) with \(B \subset A\) either \(\mu(B) = 0\) or \(\mu(B) = \mu(A)\). Note that if \(A\) is not an atom, then there is some \(B, C \in \Sigma\) such that \(B, C \subset A\) and \(\mu(B) \in (0, \frac{1}{2}\mu(A)]\) and \(\mu(C) \in [\frac{1}{2}\mu(A), \mu(A))\). A non-atomic measure is defined as a measure without atoms. Define \(\mu(\Sigma)\) as \(\{\mu(A) : A \in \Sigma\}\).

**Theorem.** If \(\mu\) is a non-atomic finite measure, then \(\mu(\Sigma) = [0, \mu(S)]\).

**Proof.** For simplicity, we may assume that \(\mu(S) = 1\), or else we could look at the non-atomic finite measure \(\frac{1}{\mu(S)} \mu\).

**Subclaim 1.** There is some \(A \in \Sigma\) such that \(\mu(A) \in (\frac{1}{4}, \frac{1}{2}]\).

**Proof of Subclaim.** We will show the stronger result that for any \(B \in \Sigma\) with \(\mu(B) > 0\), there is some \(A \in \Sigma\) with \(A \subset B\) and \(\mu(A) \in (\frac{1}{4}\mu(B), \frac{1}{2}\mu(B)]\). Let \(B \in \Sigma\) with \(\mu(B) > 0\). Assume that there is not such an \(A \in \Sigma\) with \(A \subset B\) and \(\mu(A) \in (\frac{1}{4}\mu(B), \frac{1}{2}\mu(B)]\). Let \(G = \bigcap_{n = 1}^{\infty} B_n\). Since \(G\) is closed under finite intersections, since if \(C, D \in \mathcal{C}\), then \(\mu(C \cap D) = \mu(C) + \mu(D) - \mu(C \cup D) \geq \frac{1}{4} \mu(B)\). By assumption, if \(\mu(C \cap D) \geq \frac{1}{2} \mu(B)\), then \(\mu(C \cap D) \geq \frac{3}{4} \mu(B)\), and so \(\mu(C) \geq \frac{1}{4} \mu(B)\), \(\frac{3}{2} \geq \frac{3}{4} \mu(B)\) and so by assumption, \(H \in C\) contradicting minimality of \(\alpha\).

**Subclaim 2.** For any \(B \in \Sigma\) with \(\mu(B) \in (\frac{1}{4}, \frac{1}{2}]\) and for any \(n \in \mathbb{N}\), there is some \(C \in \Sigma\) such that \(C \subset S \setminus B\) and \(\mu(C) \in [\frac{2}{4^n}, \frac{3}{2 \cdot 2^n}]\).

**Proof of Subclaim.** Use induction on \(n\). So \(\mu(S \setminus B) \in (\frac{1}{2}, \frac{3}{4})\). For \(n = 1\), we may choose \(C = S \setminus B\). Let \(n \geq 1\) and suppose there is some \(D \subset S \setminus B\) where \(\mu(D) \in [\frac{2}{4^n}, \frac{3}{2 \cdot 2^n}]\). By applying the stronger statement in the proof of the previous subclaim, there is some \(\bar{C} \subset D\) where \(\mu(\bar{C}) \in (\frac{1}{4}\mu(D), \frac{1}{2}\mu(D)]\), but since \(\mu(D) \in [\frac{2}{4^n}, \frac{3}{2 \cdot 2^n}]\), \(\mu(C) \in [\frac{2}{4^n}, \frac{3}{2 \cdot 2^n}]\). Since \(C \subset D\), \(C \subset S \setminus B\).

**Subclaim 3.** For each \(C \in \Sigma\) such that \(\frac{1}{2} < \mu(C) < \frac{1}{2}\) and for each \(n \in \mathbb{N}\), there is some \(D \in \Sigma\) such that \(C \subset D\) and \(\frac{1}{2} - \frac{3}{2 \cdot 2^n} \leq \mu(D) < \frac{1}{2}\).

**Proof of Subclaim.** Suppose \(C \in \Sigma\) such that \(\frac{1}{2} < \mu(C) < \frac{1}{2}\) and \(n \in \mathbb{N}\). We will define an increasing finite sequence of sets \(D_m\), recursively, where \(m \in \{0, 1, 2, \ldots, 4^{n-1}\}\). Let \(D_0 = C\). Suppose \(D_k\) for \(k \in \{0, 1, \ldots, m\}\) is defined and is an increasing sequence of sets. If \(\mu(D_m) + \frac{3}{2 \cdot 2^n} \geq \frac{1}{2}\), then let
Subclaim 4. There is some $E \in \Sigma$ such that $\mu(E) = \frac{1}{2}$.

Proof of Subclaim. By the first subclaim, there is some $A \in \Sigma$ such that $\mu(A) \in \left(\frac{1}{4}, \frac{1}{2}\right]$. If $\mu(A) = \frac{1}{2}$, choose $E = A$. If $\mu(A) < \frac{1}{2}$, then we define a sequence of increasing sets $B_n$ by recursion. Let $B_0 = A$. Then by the third subclaim, there is some $B_1$ such that $B_0 \subset B_1$ and $\frac{1}{2} - \frac{3}{4^n} \leq \mu(B_1) < \frac{1}{2}$, and so $\mu(B_1) \in \left(\frac{1}{4}, \frac{1}{2}\right]$. We apply the third subclaim again, to obtain some $B_2$ such that $B_1 \subset B_2$ and $\frac{1}{2} - \frac{3}{4^n} \leq \mu(B_2) < \frac{1}{2}$ and moreover $\mu(B_2) \in \left(\frac{1}{4}, \frac{1}{2}\right]$. And so on.

So $B_n$ is an increasing sequence of sets such that for each $n \in \mathbb{N}$, $\frac{1}{2} - \frac{3}{2^{2n}} \leq \mu(B_n) < \frac{1}{2}$. Let $E = \bigcup_n B_n$. Then for each $n \in \mathbb{N}$, $\frac{1}{2} - \frac{3}{2^{2n}} \leq \mu(E) \leq \frac{1}{2}$. Hence $\mu(E) = \frac{1}{2}$.

Denote the dyadic rationals in $[0, 1]$ by $\mathbb{Q}_2 := \left\{ \frac{m}{2^n} : m \in \mathbb{Z}, n \in \mathbb{N} \right\} \cap [0, 1]$.

Subclaim 5. There is a collection of subsets, $(A_r)_{r \in \mathbb{Q}_2}$, of $S$ indexed by the dyadic rationals such that $\mu(A_r) = r$ and if $r < s$, then $A_r \subset A_s$.

Proof of Subclaim. We will define the sequence of sets indexed by recursion. In the $n^{th}$ step, we will define $A_m^{2^n}$ for each $m \in \{1, 3, 5, \ldots, 2^n - 1\}$ where for $k, m \in \{0, 1, \ldots, 2^n\}$ with $k \leq m$, $A_k^{2^n} \subset A_m^{2^n}$ and $\mu(A_k^{2^n}) = \frac{k}{2^n}$.

Define $A_0 = \emptyset$ and $A_1 = S$. So $A_0 \subset A_1$.

Suppose we have define such sets $A_m^{2^n}$ for each $m \in \{0, 1, \ldots, 2^n\}$. We wish to define $A_{2m+1}^{2^n+1}$ for each $k \in \{0, 1, \ldots, 2^n - 1\}$.

Let $k \in \{0, 1, \ldots, 2^n - 1\}$ and let $C = A_{k+1}^{2^n+1} \setminus A_k^{2^n}$.

The measure $\nu = 2^n \cdot \mu | C$ is a non-atomic measure with $\nu(C) = 1$. So there is some $E \subset C$ such that $\nu(E) = \frac{1}{2}$ by the fourth subclaim. Let $A_{2m+1}^{2^n+1} = A_m^{2^n} \cup E$. Then $\mu(A_{2m+1}^{2^n+1}) = \frac{m}{2^n} + \frac{1}{2^n} \cdot \frac{1}{2} = \frac{2m+1}{2^{n+1}}$ and $A_m^{2^n} \subset A_{2^{n+1}+1}^{2^n+1} \subset A_{2m+1}^{2^n+1}$.

Now we can finish the proof to the theorem.

We have defined for each $\beta \in \mathbb{Q}_2$, a set $A_\beta$ such that $\mu(A_\beta) = \beta$ and for each $\beta, \gamma \in \mathbb{Q}_2$ with $\beta < \gamma$, $A_\beta \subset A_\gamma$.

For each $c \in [0, 1]$, define a set $B_c = \bigcup_{\beta < c} A_\beta$. (Note: This union is countable and so $B_c \in \Sigma$).

Also for any $\beta, \gamma \in \mathbb{Q}_2$ with $\beta \leq c \leq \gamma$, $A_\beta \subset B_c \subset A_\gamma$ and so $\beta \leq \mu(B_c) \leq \gamma$.

Hence $\mu(B_c) = c$.\