What is the Veronese map?

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Abstract

We introduce the Veronese map $\nu_d : \mathbb{P}^n \to \mathbb{P}^{N(n,d)}$, a popular example in algebraic geometry. We consider the d=2 and n=2 case as the Veronese surface. The Veronese variety is also defined and for the n=2, d=2 case shown equivalent to a determinantal variety. We state the map's embedding property. A consequence of this is the isomorphism of $\mathbb{P}^n$ with $\nu_d(\mathbb{P}^n)$ and the Veronese variety.

1 Preliminary Definitions

Definition 1. (Projective space) $\mathbb{P}^n(\mathbb{C}) = \{[Z_0 \cdots Z_n] \mid (\lambda Z_0, ...\lambda Z_n) = (Z_0, \cdots Z_n) \in \mathbb{C}^{n+1} - \{0\} \text{ for } \lambda \in \mathbb{C}^*\} = \mathbb{P}^n$

Remark 1. Notice $P^n(\mathbb{C})$= Gr(1,n+1)= space of lines through the origin in $\mathbb{C}^{n+1}$, a Grassmanian. Also notice that instead of $\mathbb{C}$ we have could also used any algebraically closed field.

Definition 2. (Projective Variety) X $\subset \mathbb{P}$ is the zero locus of a finite set of homogeneous polynomials that generate a prime ideal.
Recall a homogeneous polynomial F of degree d on $\mathbb{C}^{n+1}$ means: F$(\lambda Z_0, \cdots, \lambda Z_n)$ $= \lambda^d F(Z_0 \cdots Z_n)$.
This allows us to define zero loci $\subset \mathbb{P}^n$ for a collection of homogeneous polynomials.

Remark 2. We will use the notation Z$(\{f_\alpha\}_\alpha)$ to denote the common zero locus of a set of polynomials $\{f_\alpha\}_\alpha$.

2 The Veronese map

Definition 3. (Veronese map) the Veronese map of degree d is $\nu_d : \mathbb{P}^n \to \mathbb{P}^N$ as: $[X_0 \cdots X_n] \mapsto [: \cdots X_0^{i_0} \cdots X_n^{i_n} : \cdots] = [\cdots Z_{i_0 \cdots i_n}(X_0, \cdots X_n), \cdots] = [Z_0 \cdots Z_N]$ where $X_0^{i_0} \cdots X_n^{i_n}$ ranges over all monomials of degree d in $X_0 \cdots X_n$, meaning $i_0 + \cdots + i_n =$ d, for *some* ordering.

Remark 3. Notice $\{[\cdots Z_{i_0 \cdots i_n}(X_0, \cdots X_n), \cdots]\} \subset \{[Z_0 \cdots Z_N]\}$

Proposition 1. N= $\binom{n+d}{d}$ -1

Proof. Count the number of monomials of degree d using d balls and n walls then subtract one by the projective space definition.

$\square$

Remark 4. Notice that given N and d, it is generally complicated to find n. Similarly given N and n, it is generally complicated to find d.

3 The n=2, d=2 case

Definition 4 (Veronese surface)**.** let n=2, d=2
$\nu_2 : \mathbb{P}^2 \to \mathbb{P}^5$
$\nu_2([X_0, X_1, X_2]) = [X_0X_0, X_1X_1, X_2X_2, X_0X_1, X_0X_2, X_1X_2]$.
The Veronese surface is $\nu_2(\mathbb{P}^2)$.

Definition 5 (Veronese variety for n=2, d= 2)**.** $\mathbb{V}_{2,2}$:= $Z(\{Z_0Z_1-Z_3Z_3, Z_0Z_5-Z_4Z_3, Z_0Z_2-Z_4Z_4, Z_3Z_5-Z_4Z_1, Z_3Z_2-Z_4Z_5, Z_1Z_2-Z_5Z_5\})$ where $[Z_0 \cdots Z_5]$ $\in \mathbb{P}^5$

Remark 5. Notice that $\mathbb{V}_{2,2}$ = Z(ker(θ)) for the n=2, d=2 case of definition 6. We do not prove this fact.
Thus $\mathbb{V}_{2,2}$ is a projective variety by proposition 3.

Lemma 1 (equivalent condition for p $\times$ q matrix to be rank k)**.** Given a p$\times$q matrix M; WLOG p$\leq$ q. Let k $\geq$ 0, k+1 $\leq$ p. All (k+1)$\times$(k+1) minors are 0 and there is a k$\times$k non zero minor iff rank(M) = k. The case of rank(M)=p is trivial.

Proof. Fact: There exists a j$\times$j nonzero minor iff there exists j linearly independent rows.

There is a nonzero minor of order k iff there must be at least k linearly independent rows in M. iff rank(M) $\geq$ k.

rank(M) $\geq$ k+1 iff there exists at least k+1 linearly independent rows. meaning there is an k+1 order submatrix taken from these k+1 rows that is invertible and thus has nonzero determinant iff there is some k+1 order minor nonzero. Therefore by contrapositive: rank(M) $\leq$ k iff all k+1 order minors are zero.

Therefore rank(M)=k iff all (k+1)$\times$(k+1) minors are 0 and there is a k$\times$k non zero minor $\square$

Exercise 1. Show that the dimension of the row space = dimension of the column space for any p$\times$ q matrix. (This proves one direction of the Fact in the above proof.)

Proposition 2. $\mathbb{V}_{2,2} =$
$\{[Z_0 \cdots Z_5] \in \mathbb{P}^5 \mid \begin{pmatrix} Z_0 & Z_3 & Z_4 \\ Z_3 & Z_1 & Z_5 \\ Z_4 & Z_5 & Z_2 \end{pmatrix} \text{ of rank 1 } \} := \mathbb{D}_{2,2}$

Proof. (The LHS is the locus generated by all the 2×2 minors of M= $\begin{pmatrix} Z_0 & Z_3 & Z_4 \\ Z_3 & Z_1 & Z_5 \\ Z_4 & Z_5 & Z_2 \end{pmatrix}$ and at least one entry of M is nonzero since $[Z_0 \cdots Z_5] \in \mathbb{P}^5$) iff M has rank 1 by lemma 1 with k=1, p=q=3. □

We mention a corollary of the the previous proposition and the theorem on the Veronese map being an embedding:

Corollary 1 (Equivalent definitions)**.** For n= 2, d= 2, $\mathbb{P}^2 \cong \nu_2(\mathbb{P}^2) \cong \mathbb{V}_{2,2} \cong \mathbb{D}_{2,2}$

Proof. See theorem 1 (the Veronese map is an embedding to the general Veronese Variety: ($\mathbb{P}^n \cong \nu_2(\mathbb{P}^2) \cong \mathbb{V}_{2,2}$) and proposition 2 ($\mathbb{V}_{2,2} \cong \mathbb{D}_{2,2}$) □

4 The general case

Definition 6 (the General Veronese Variety)**.** Let $\theta : \mathbb{C}[\{Z_{i_0 \ldots i_n}\}] \to \mathbb{C}[X_0 \cdots X_n]$ where $i_0 + \cdots i_n =$ d and where θ: $\sum \prod \cdots Z_{i_0 \cdots i_n}(X_0, \cdots X_n), \cdots \mapsto \sum \prod \cdots X_0^{i_0} \cdots X_n^{i_n} \cdots$

$\{Z_{i_0 \ldots i_n}\}$ are the coordinates of $\mathbb{P}^N$ in terms of the Veronese map. On the other hand, the codomain is in the X_j, the standard coordinates of $\mathbb{P}^n$
(that's why θ is *not* the identity map!).

Define the Veronese variety $\mathbb{V}_{n,d}$:= Z(ker(θ))

Exercise 2. Check that θ is a ring homomorphism.

Theorem 1 (Veronese Map is an Embedding to the general Veronese Variety)**.**
X= $\mathbb{P}^n$ a projective variety and Y= $\nu_d(X) \subset \mathbb{P}^N$, X and Y are isomorphic to the general Veronese Variety $\mathbb{V}_{n,d}$

Remark 6. By X isomorphic with Y, we mean that there exists a biregular map between X and Y. See [3] for definition of a regular map; Hartshorne calls regular *maps* morphisms.

Remark 7. In fact the more general fact is true: X $\subset \mathbb{P}^n$ is a projective variety then ν_d(X)=Y $\cong$ X with X and Y not necessarily equal to $\mathbb{V}_{n,d}$ and Y a projective variety.

Proposition 3. $\mathbb{V}_{n,d}$ is a projective variety

Proof. $\ker(\theta)$ is clearly an ideal. Since $\mathbb{C}[X_0 \cdots X_n]$ is an integral domain and $\mathbb{C}[\{Z_{i_0 \dots i_n}\}]/\ker(\theta) \cong$ some subring of $\mathbb{C}[X_0 \cdots X_n]$, $\ker(\theta)$ is a prime ideal.

Let f $\in \mathbb{C}[\{Z_{i_0 \dots i_n}\}]$, f= $\sum_{j=0}^{k} f_j$ be a finite sum of homogeneous polynomials of degree i. To show that $\ker(\theta)$ is generated by homogeneous polynomials it suffices to show f $\in \ker(\theta)$ iff $f_j \in \ker(\theta)$ for j= 0 $\cdots$ k.
If θ(f) =0 $= \theta(\sum_{j=0}^{k} f_j) = \sum_{j=0}^{k} \theta(f_j)$
and since $\theta(f_j)$ is homogeneous of degree d·j, (why?)
then we cannot have cancellation in the sum and so each $\theta(f_j)$=0
iff $f_j \in \ker(\theta)$ for all j = 0 $\cdots$ k.

Clearly $\ker(\theta)$ is *finitely* generated by Hilbert basis theorem.
Thus we can pick a finite generating set of polynomials FG= $\{f\}$ and find their homogenous components f_j of degree j such that $\sum_{j=0}^{k} f_j = f$ for every f $\in$ FG. These f_j for all f $\in$ FG are a finite homogeneous generating set of $\ker(\theta)$ □

Remark 8. Notice: Z($\ker(\theta)$) is the zero locus of homogeneous polynomials $Z_I Z_J - Z_K Z_L$ with I+J= K+L as finite sequences.

Exercise 3. Show that $Z(<\{f\}>) = Z(\{f\})$ where $<\{f\}>$ means *ideal* generated by $\{f\}$.

Exercise 4. What do some elements of $\{Z_I Z_J - Z_K Z_L\}$ look like for n=2, d=2? hint: consider definition 5

Remark 9. I originally wrote the proof of the embedding theorem and a couple of propositions about properties of the Veronese map. All such writings have been removed due to time constraints.

References

[1] Griffiths, P. and Harris, J. Principles of Algebraic Geometry, Wiley and Sons, 1978.

[2] Harris, J. Algebraic Geometry, a First Course, Graduate Texts in Mathematics, Springer-Verlag, 1992.

[3] Hartshorne, R. Algebraic Geometry, Graduate Texts in Mathematics, Springer-Verlag, 1977.