What is the Veronese map?

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Abstract

We introduce the Veronese map $\nu_d : \mathbb{P}^n \rightarrow \mathbb{P}^{N(n,d)}$, a popular example in algebraic geometry. We consider the $d=2$ and $n=2$ case as the Veronese surface. The Veronese variety is also defined and for the $n=2$, $d=2$ case shown equivalent to a determinantal variety. We state the map's embedding property. A consequence of this is the isomorphism of $\mathbb{P}^n$ with $\nu_d(\mathbb{P}^n)$ and the Veronese variety.

1 Preliminary Definitions

Definition 1. (Projective space) $\mathbb{P}^n(\mathbb{C}) = \{ [Z_0 : \cdots : Z_n] \ | \ (\lambda Z_0, \cdots, \lambda Z_n) = (Z_0, \cdots, Z_n) \in \mathbb{C}^{n+1} - \{0\} \}$ for $\lambda \in \mathbb{C}^*$.

Remark 1. Notice $\mathbb{P}^n(\mathbb{C}) = \text{Gr}(1,n+1)$, the space of lines through the origin in $\mathbb{C}^{n+1}$, a Grassmanian. Also notice that instead of $\mathbb{C}$ we could have used any algebraically closed field.

Definition 2. (Projective Variety) $X \subset \mathbb{P}$ is the zero locus of a finite set of homogeneous polynomials that generate a prime ideal.

Recall a homogeneous polynomial $F$ of degree $d$ on $\mathbb{C}^{n+1}$ means: $F(\lambda Z_0, \cdots, \lambda Z_n) = \lambda^d F(Z_0, \cdots, Z_n)$.

This allows us to define zero loci $\subset \mathbb{P}^n$ for a collection of homogeneous polynomials.

Remark 2. We will use the notation $Z(\{f_\alpha\}_\alpha)$ to denote the common zero locus of a set of polynomials $\{f_\alpha\}_\alpha$.

2 The Veronese map

Definition 3. (Veronese map) the Veronese map of degree $d$ is $\nu_d : \mathbb{P}^n \rightarrow \mathbb{P}^N$ as:

$[X_0 \cdots X_n] \mapsto [\cdots X^i_0 \cdots X^i_n : \cdots] = \{ \cdots Z_{i_0 \cdots i_n}(X_0, \cdots, X_n), \cdots \} = [Z_0 \cdots Z_N]$

where $X^i_0 \cdots X^i_n$ ranges over all monomials of degree $d$ in $X_0 \cdots X_n$, meaning $i_0 + \cdots + i_n = d$, for some ordering.

Remark 3. Notice $\{ [\cdots Z_{i_0 \cdots i_n}(X_0, \cdots, X_n), \cdots] \} \subset \{ [Z_0 \cdots Z_N] \}$
Proposition 1. \( N = \binom{n+d}{d} - 1 \)

Proof. Count the number of monomials of degree \( d \) using \( d \) balls and \( n \) walls then subtract one by the projective space definition.

Remark 4. Notice that given \( N \) and \( d \), it is generally complicated to find \( n \). Similarly given \( N \) and \( n \), it is generally complicated to find \( d \).

3 The \( n=2, d=2 \) case

Definition 4 (Veronese surface). let \( n=2, d=2 \)
\[
\nu_2 : \mathbb{P}^2 \to \mathbb{P}^5
\]
\[
\nu_2([X_0, X_1, X_2]) = [X_0X_0, X_1X_1, X_2X_2, X_0X_1, X_0X_2, X_1X_2].
\]
The Veronese surface is \( \nu_2(\mathbb{P}^2) \).

Definition 5 (Veronese variety for \( n=2, d=2 \)). \( \nu_{2,2} := Z(\{Z_0Z_1 - Z_3Z_3, Z_0Z_5 - Z_4Z_3, Z_0Z_2 - Z_4Z_4, Z_3Z_5 - Z_4Z_1, Z_2Z_2 - Z_4Z_5, Z_1Z_2 - Z_5Z_5\}) \) where \([Z_0 \cdots Z_5] \in \mathbb{P}^5\)

Remark 5. Notice that \( \nu_{2,2} = Z(\ker(\theta)) \) for the \( n=2, d=2 \) case of definition 6.
We do not prove this fact.
Thus \( \nu_{2,2} \) is a projective variety by proposition 3.

Lemma 1 (equivalent condition for \( p \times q \) matrix to be rank \( k \)). Given a \( p \times q \) matrix \( M \); WLOG \( p \leq q \).
Let \( k \geq 0 \), \( k+1 \leq p \). All \( (k+1) \times (k+1) \) minors are 0 and there is a \( k \times k \) non-zero minor iff \( \text{rank}(M) = k \).
The case of \( \text{rank}(M)=p \) is trivial.

Proof. Fact: There exists a \( j \times j \) nonzero minor iff there exists \( j \) linearly independent rows.

There is a nonzero minor of order \( k \) iff there must be at least \( k \) linearly independent rows in \( M \).
iff \( \text{rank}(M) \geq k \).

\( \text{rank}(M) \geq k+1 \) iff there exists at least \( k+1 \) linearly independent rows. meaning there is an \( k+1 \) order submatrix taken from these \( k+1 \) rows that is invertible and thus has nonzero determinant iff there is some \( k+1 \) order minor nonzero. Therefore by contrapositive: \( \text{rank}(M) \leq k \) iff all \( k+1 \) order minors are zero.

Therefore \( \text{rank}(M)=k \) iff all \( (k+1) \times (k+1) \) minors are 0 and there is a \( k \times k \) non zero minor

Exercise 1. Show that the dimension of the row space = dimension of the column space for any \( p \times q \) matrix. (This proves one direction of the Fact in the above proof.)
Proposition 2. $\mathbb{V}_{2,2} = \{ [Z_0 \cdots Z_5] \in \mathbb{P}^5 \mid \begin{pmatrix} Z_0 & Z_3 & Z_4 \\
Z_3 & Z_1 & Z_5 \\
Z_4 & Z_5 & Z_2 \end{pmatrix} \text{ of rank } 1 \} := \mathbb{D}_{2,2}$

Proof. (The LHS is the locus generated by all the $2 \times 2$ minors of $M = \begin{pmatrix} Z_0 & Z_3 & Z_4 \\
Z_3 & Z_1 & Z_5 \\
Z_4 & Z_5 & Z_2 \end{pmatrix}$ and at least one entry of $M$ is nonzero since $[Z_0 \cdots Z_5] \in \mathbb{P}^5$) iff $M$ has rank 1 by lemma 1 with $k=1$, $p=q=3$. □

We mention a corollary of the the previous proposition and the theorem on the Veronese map being an embedding:

Corollary 1 (Equivalent definitions). For $n = 2$, $d = 2$, $\mathbb{P}^2 \cong \nu_2(\mathbb{P}^2) \cong \mathbb{V}_{2,2} \cong \mathbb{D}_{2,2}$

Proof. See theorem 1 (the Veronese map is an embedding to the general Veronese Variety: $(\mathbb{P}^n \cong \nu_2(\mathbb{P}^2) \cong \mathbb{V}_{2,2})$ and proposition 2 ($\mathbb{V}_{2,2} \cong \mathbb{D}_{2,2}$) □

4. The general case

Definition 6 (the General Veronese Variety). Let $\theta : \mathbb{C}[[\{Z_{i_0 \cdots i_n}\}]] \to \mathbb{C}[X_0 \cdots X_n]$ where $i_0 + \cdots + i_n = d$ and where $\theta : \sum \prod \cdots Z_{i_0 \cdots i_n} (X_0, \cdots, X_n) \mapsto \sum \prod X_{i_0} \cdots X_{i_n} \cdots$

{[$Z_{i_0 \cdots i_n}$]} are the coordinates of $\mathbb{P}^N$ in terms of the Veronese map. On the other hand, the codomain is in the $X_j$, the standard coordinates of $\mathbb{P}^n$ (that’s why $\theta$ is not the identity map!).

Define the Veronese variety $\mathbb{V}_{n,d} := \mathbb{V}($ker($\theta))$

Exercise 2. Check that $\theta$ is a ring homomorphism.

Theorem 1 (Veronese Map is an Embedding to the general Veronese Variety). $\mathbb{X} = \mathbb{P}^n$ a projective variety and $\mathbb{Y} = \mathbb{V}_d(\mathbb{X}) \subset \mathbb{P}^N$, $\mathbb{X}$ and $\mathbb{Y}$ are isomorphic to the general Veronese Variety $\mathbb{V}_{n,d}$

Remark 6. By $\mathbb{X}$ isomorphic with $\mathbb{Y}$, we mean that there exists a biregular map between $\mathbb{X}$ and $\mathbb{Y}$. See [3] for definition of a regular map; Hartshorne calls regular maps morphisms.

Remark 7. In fact the more general fact is true: $\mathbb{X} \subset \mathbb{P}^n$ is a projective variety then $\nu_d(\mathbb{X}) = \mathbb{Y} \cong \mathbb{X}$ with $\mathbb{X}$ and $\mathbb{Y}$ not necessarily equal to $\mathbb{V}_{n,d}$ and $\mathbb{Y}$ a projective variety.

Proposition 3. $\mathbb{V}_{n,d}$ is a projective variety
Proof. \( \ker(\theta) \) is clearly an ideal. Since \( \mathbb{C}[X_0 \cdots X_n] \) is an integral domain and \( \mathbb{C}[[Z_{i_0} \cdots Z_{i_n}]]/\ker(\theta) \cong \) some subring of \( \mathbb{C}[X_0 \cdots X_n] \), \( \ker(\theta) \) is a prime ideal.

Let \( f \in \mathbb{C}[[Z_{i_0} \cdots Z_{i_n}]] \), \( f = \sum_{j=0}^{k} f_j \) be a finite sum of homogeneous polynomials of degree \( i \). To show that \( \ker(\theta) \) is generated by homogeneous polynomials it suffices to show \( f \in \ker(\theta) \) iff \( f_j \in \ker(\theta) \) for \( j = 0 \cdots k \).

If \( \theta(f) = 0 = \theta(\sum_{j=0}^{k} f_j) = \sum_{j=0}^{k} \theta(f_j) \)

and since \( \theta(f_j) \) is homogeneous of degree \( d_j \), (why?)

then we cannot have cancellation in the sum and so each \( \theta(f_j) = 0 \)

iff \( f_j \in \ker(\theta) \) for all \( j = 0 \cdots k \).

Clearly \( \ker(\theta) \) is finitely generated by Hilbert basis theorem.
Thus we can pick a finite generating set of polynomials \( \text{FG} = \{ f \} \) and find their homogenous components \( f_j \) of degree \( j \) such that \( \sum_{j=0}^{k} f_j = f \) for every \( f \in \text{FG} \).
These \( f_j \) for all \( f \in \text{FG} \) are a finite homogeneous generating set of \( \ker(\theta) \)

Remark 8. Notice: \( Z(\ker(\theta)) \) is the zero locus of homogeneous polynomials \( Z_I Z_J - Z_K Z_L \) with \( I+J = K+L \) as finite sequences.

Exercise 3. Show that \( Z(< \{ f \} >) = Z(\{ f \}) \) where \( < \{ f \} > \) means ideal generated by \( \{ f \} \).

Exercise 4. What do some elements of \( \{ Z_I Z_J - Z_K Z_L \} \) look like for \( n=2, d=2 \)?

Remark 9. I originally wrote the proof of the embedding theorem and a couple of propositions about properties of the Veronese map. All such writings have been removed due to time constraints.

References

