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Classification of Certain Compact Riemannian Manifolds with Harmonic Curvature and Non-parallel Ricci Tensor

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1. Introduction

For any Riemannian manifold, the divergence δR of its curvature tensor R satisfies the well-known identity $\delta R = dS$, i.e., in local coordinates

$$\nabla^i R_{hijk} = \nabla_k S_{hj} - \nabla_j S_{hk}, \tag{1}$$

S being the Ricci tensor. While every manifold with parallel Ricci tensor has *harmonic curvature*, i.e., satisfies $\delta R = 0$, there are examples ([3], Theorem 5.2) of open Riemannian manifolds with $\delta R = 0$ and $\nabla S \neq 0$. In [1] Bourguignon has asked the question whether the Ricci tensor of a *compact* Riemannian manifold with harmonic curvature must be parallel.

The aim of this paper is to give examples (see Remark 2) answering this question in the negative. All our examples are conformally flat (Corollary 1). Moreover, we obtain some classification results, restricting our consideration to Riemannian manifolds with $\delta R = 0$, $\nabla S \neq 0$ and such that the Ricci tensor S has at any point less than three distinct eigenvalues. Starting from a description of their local structure at generic points (Theorem 1), we find all four-dimensional, analytic, complete and simply connected manifolds of this type (Theorem 2). They are all non-compact, but some of them do possess compact quotients. Next we prove (Theorem 3) that all compact four-dimensional analytic Riemannian manifolds with the above properties are covered by $S^1 \times S^3$ with a metric of an explicitly described form.

Throughout this paper, by a manifold we mean a connected paracompact manifold of class C^∞ or analytic. By abuse of notation, concerning Riemannian manifolds we often write M instead of (M, g) and $\langle u, v \rangle$ instead of $g(u, v)$ for tangent vectors u, v.

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2. Preliminaries

The results and arguments of this section are similar to those of Élie Cartan's work on isoparametric surfaces (cf. [2]). The author is obliged to Professor James Eells for pointing this out to him.

Let M be a Riemannian manifold of class C^∞ or analytic. Consider a *Codazzi tensor* A on M, i.e., a symmetric $(0,2)$-tensor field, satisfying the Codazzi equation $(\nabla_u A)(v,w)=(\nabla_v A)(u,w)$ for arbitrary tangent vectors u, v, w. In view of (1), a Riemannian manifold has harmonic curvature if and only if its Ricci tensor is Codazzi.

Define the integer-valued function E_A on M by $E_A(x)=$(the number of distinct eigenvalues of A_x) and set $M_A=\{x\in M\colon E_A \text{ is constant in a neighbourhood of } x\}$. Clearly, M_A is an open dense subset of M. In each connected component of M_A, the eigenvalues of A are well-defined and everywhere distinct differentiable (resp. analytic) functions, called in the sequel the *eigenfunctions* of A, and the eigenspaces of A form mutually orthogonal differentiable (resp. analytic) distributions (eigendistributions of A). We have

Lemma 1. *Let A be a Codazzi tensor on a Riemannian manifold M. Then, in each connected component of M_A,*

(i) *If two (locally defined) vector fields u,v satisfy $Au=\lambda u$, $Av=\lambda v$, i.e., belong to the eigendistribution V_λ corresponding to the eigenfunction λ of A, then*

$$A\nabla_v u=\lambda\nabla_v u+(v\lambda)u-\langle u,v\rangle\nabla\lambda. \tag{2}$$

(ii) *Given distinct eigenfunctions λ,μ of A and (local) vector fields $v\in V_\lambda$, $u\in V_\mu$ with $|u|=1$, we have*

$$v\mu=(\mu-\lambda)\langle\nabla_u u,v\rangle. \tag{3}$$

Proof. (i) For any vector field w, the Leibniz rule implies

$$\langle A\nabla_v u,w\rangle=\langle\nabla_v(\lambda u)-(\nabla_v A)u,w\rangle=\langle(v\lambda)u+\lambda\nabla_v u,w\rangle-(\nabla_w A)(u,v)$$

and

$$\begin{aligned}(\nabla_w A)(u,v)&=w\langle Au,v\rangle-\langle\nabla_w u,Av\rangle-\langle Au,\nabla_w v\rangle\\&=(w\lambda)\langle u,v\rangle+\lambda[w\langle u,v\rangle-\langle\nabla_w u,v\rangle-\langle u,\nabla_w v\rangle]\\&=\langle\langle u,v\rangle\nabla\lambda,w\rangle,\end{aligned}$$

as required.

(ii) Clearly, $\langle u,v\rangle=0$, hence the Codazzi equation yields

$$\begin{aligned}(\mu-\lambda)\langle\nabla_u u,v\rangle&=-\lambda\langle\nabla_u u,v\rangle-\mu\langle u,\nabla_u v\rangle=-A(\nabla_u u,v)-A(u,\nabla_u v)\\&=(\nabla_u A)(u,v)=(\nabla_v A)(u,u)=v\langle Au,u\rangle=v\mu.\end{aligned}$$

This completes the proof.

Lemma 2. *Given a Codazzi tensor A on a Riemannian manifold M, we have, in each connected component of M_A,*

(i) *The eigendistributions of A are integrable and their leaves are totally umbilic submanifolds of M.*

(ii) *Each eigenfunction λ of multiplicity greater than one is constant along the leaves of the corresponding eigendistribution V_λ.*

Proof. Given an eigenfunction λ with $\dim V_\lambda \geqq 2$ and a fixed (local) unit vector field $v \in V_\lambda$, choose a local unit field $u \in V_\lambda$ with $\langle u, v\rangle = 0$. By (2), $v\lambda = \langle \lambda \nabla_v u + (v\lambda) u - \langle u, v\rangle \nabla\lambda, u\rangle = A(\nabla_v u, u) = \langle \nabla_v u, \lambda u\rangle = 0$. This implies (ii) if we know that the eigendistributions are integrable. To prove this, consider $u, v \in V_\lambda$ and assume, without loss of generality, that $\dim V_\lambda \geqq 2$. Hence $u\lambda = v\lambda = 0$ and (2) yields $A[v,u] = A(\nabla_v u - \nabla_u v) = \lambda[v,u]$, i.e., $[v,u] \in V_\lambda$, as required. In order to show that the leaves of V_λ are totally umbilic, fix a local unit vector field $u \in V_\lambda$ and a vector field v normal to V_λ. The second fundamental form of the leaves of V_λ with respect to v is given by $b^v(u,u) = -\langle \nabla_u u, v\rangle$. Asssuming, without loss of generality, that $v \in V_\mu$, $\mu \neq \lambda$, we obtain from (3) $b^v(u,u) = (\mu - \lambda)^{-1} v\lambda$, i.e., $b^v_x(u,u)$ is independent of the unit vector u tangent to the leaf at x, which completes the proof.

For a later application, we also need

Lemma 3. *Let A be a Codazzi tensor on an n-dimensional Riemannian manifold M, $n \geqq 3$. Suppose U is a connected open subset of M_A such that trace A is constant in U and $\nabla A \neq 0$ at some point of U. If A has exactly two eigenfunctions λ, μ in U, and if $\dim V_\lambda \leqq \dim V_\mu$, then $\dim V_\lambda = 1$, the integral curves of V_λ are geodesics and every leaf of V_μ has constant mean curvature.*

Proof. Given a vector field $u \in V_\mu$, (ii) of Lemma 2 yields $u\mu = 0$ and

$$u\lambda = (\dim V_\lambda)^{-1} u[\operatorname{trace} A - (\dim V_\mu)\mu] = 0. \tag{4}$$

If we had $\dim V_\lambda \geqq 2$, then, by (4) and (ii) of Lemma 2, λ would be constant and hence so would be μ. Formula (2) applied to any vector fields $u, w \in V_\lambda$ (resp. $u, w \in V_\mu$) yields then $\nabla_w u \in V_\lambda$ (resp. $\nabla_w u \in V_\mu$), so that the leaves of both eigendistributions would be totally geodesic. Suppose now that $u \in V_\lambda$ and $w \in V_\mu$, i.e., w is normal to V_λ. Hence so is $\nabla_u w$ by the total geodesy of V_λ. Thus, $\nabla_u w \in V_\mu$ whenever $w \in V_\mu$, for any u, i.e., V_μ is a parallel distribution. The local de Rham decomposition theorem implies $\nabla A = 0$ in U, which is a contradiction. Therefore $\dim V_\lambda = 1$.

Now fix a local unit vector field $v \in V_\lambda$. Clearly, $\langle \nabla_v v, v\rangle = 0$ and, in view of (3) and (4), $\langle \nabla_v v, u\rangle = 0$ for any $u \in V_\mu$. Hence $\nabla_v v = 0$, i.e., V_λ is geodesic. Finally, the mean curvature of the leaves of V_μ is given by $\zeta = (\lambda - \mu)^{-1} v\mu$ (cf. proof of Lemma 2). Next, for a vector field $u \in V_\mu$, (4) implies $u\zeta = (\lambda - \mu)^{-1} uv\mu$, while $uv\mu = [u,v]\mu$ and $\langle [u,v], v\rangle = -\langle \nabla_v u, v\rangle = \langle u, \nabla_v v\rangle = 0$. Thus, $[u,v] \in V_\mu$, which yields $[u,v]\mu = 0$ by (ii) of Lemma 2. This completes the proof.

3. The Local Structure

It will be convenient to consider the following construction (cf. [5]). Let (M, h^M) and (N, h^N) be Riemannian manifolds, $F\colon M \to \mathbb{R}$ a positive function. Define the

F-warped product $M\times_F N$ of M and N to be the Riemannian manifold $(M\times N, h^M\times_F h^N)$ with

$$(h^M\times_F h^N)_{(x,y)}(u+X, v+Y)=h^M_x(u,v)+F(x)\,h^N_y(X,Y)$$

for $u,v\in T_xM$, $X,Y\in T_yN$.

Remark 1. Since we are particularly interested in warped products $M\times_F N$ with $\dim M=1$, it is useful to write down the local coordinate expressions for some geometric quantities in this case. In a suitable product chart $t=x^0, x^1,\dots,x^{n-1}$ for $M\times N$ we have, putting $h=h^N$, $g=h^M\times_F h^N$ and $q=\log F$, $g_{00}=1$, $g_{0i}=0$, $g_{ij}=e^q h_{ij}$, $\Gamma^0_{00}=\Gamma^0_{0i}=\Gamma^i_{00}=0$, $\Gamma^0_{ij}=-\frac{1}{2}e^q q' h_{ij}$, $\Gamma^i_{0j}=\frac{1}{2}q'\delta^i_j$, $\Gamma^i_{jk}=H^i_{jk}$. Here and in the sequel i,j,k run through $\{1,\dots,n-1\}$, the prime stands for d/dt, while the Γ's (resp. H's) are the Christoffel symbols of g (resp. of h with respect to the chart $x^1,\dots,x^{n-1}$ of N). Furthermore, denoting by ∇ and S (resp. by D and ρ) the Riemannian connection and the Ricci tensor of g (resp. of h), we have

$$\begin{aligned} S_{00}&=\frac{1-n}{4}[2q''+(q')^2], \qquad S_{0i}=0,\\ S_{ij}&=\rho_{ij}-\tfrac{1}{4}e^q[2q''+(n-1)(q')^2]\,h_{ij}, \end{aligned}\tag{5}$$

$$\begin{aligned} \nabla_0 S_{00}&=\frac{1-n}{2}[q'''+q'q''], \qquad \nabla_0 S_{i0}=\nabla_i S_{00}=0,\\ \nabla_0 S_{ij}&=-q'\rho_{ij}-\tfrac{1}{2}e^q[q'''+(n-1)q'q'']\,h_{ij},\\ \nabla_i S_{0j}&=-\tfrac{1}{2}q'\rho_{ij}+\frac{2-n}{4}e^q q' q'' h_{ij}, \end{aligned}\tag{6}$$

and

$$\nabla_k S_{ij}=D_k\rho_{ij}.\tag{7}$$

Lemma 4. *Let M be an interval of $\mathbb{R}$, considered with its standard metric, $F\colon M\to\mathbb{R}$ a non-constant positive C^∞ function and N an $(n-1)$-dimensional Riemannian manifold, $n\geqq 3$. Then the following conditions are equivalent:*

(i) *$M\times_F N$ has harmonic curvature tensor;*

(ii) *N is an Einstein space and the positive function $\varphi=F^{n/4}\colon M\to\mathbb{R}$ satisfies the ordinary differential equation*

$$\varphi''-\frac{n\kappa}{4(n-1)}\varphi^{1-4/n}=p\varphi\tag{8}$$

for some real number p, κ being the constant scalar curvature of N.

Proof. Let $M\times_F N$ have harmonic curvature. If $n=3$, then (1) and (7) yield $D_k\rho_{ij}=D_j\rho_{ik}$, which implies that the 2-dimensional manifold N has constant curvature. For $n\geqq 4$, (1) and (6) together with the non-constancy of q imply that ρ is a multiple of h. Therefore N is Einstein. Denoting by κ its constant scalar curvature, it is clear from (1), (6) and (7) that $M\times_F N$ satisfies $\delta R=0$ if and only if $q'''+\frac{n}{2}q'q''+\frac{\kappa}{n-1}q'e^{-q}=0$, i.e., $q''+\frac{n}{4}(q')^2-\frac{\kappa}{n-1}e^{-q}=\frac{4}{n}p$ for some real p. This is equivalent to (8) with $\varphi = e^{nq/4}$. The implication (ii) → (i) is now obvious, which completes the proof.

Remark 2. Lemma 4 already gives examples of compact Riemannian four-manifolds with harmonic curvature and non-parallel Ricci tensor. To this end, consider $M \times_F N$, M being the unit circle (instead of an interval) and N a compact 3-manifold of constant sectional curvature $+1$ (i.e., $\kappa = 6$), where, for instance, $F(t) = 2 + \cos t$. The Ricci tensor is not parallel since $\nabla_0 S_{00} \neq 0$. Examples of this type are contained in a more general classification given in Sect. 4.

We are now in a position to prove the following local structure theorem.

Theorem 1. *Let (M, g) be an n-dimensional Riemannian manifold with harmonic curvature tensor, $n \geqq 3$. Suppose x is a point of M such that $(\nabla S)_x \neq 0$ and $x \in M_S$, $E_S(x) = 2$, i.e., in a neighbourhood of x, the Ricci tensor has exactly two eigenvalues. Then*

(i) *A certain neighbourhood of x is isometric to a warped product $I \times_F V$, I being an interval of $\mathbb{R}$, V an $(n-1)$-dimensional Einstein space with scalar curvature κ and $F: I \to \mathbb{R}$ a non-constant positive function such that $\varphi = F^{n/4}$, viewed as a function of the arc length parameter, is a solution of* (8).

On the other hand, given an $(n-1)$-dimensional Einstein space V $(n \geqq 3)$ with scalar curvature κ and a positive function F on an interval I, such that $\varphi = F^{n/4}$ satisfies (8), *the warped product $I \times_F V$ has harmonic curvature.*

(ii) *If, moreover, $n = 4$ and (M, g) is analytic and geodesically complete, then, in the notations of* (i), *V is a space of constant sectional curvature $K = \frac{\kappa}{6}$ and F is given by one of the following formulae*

$$F(t) = Kt^2 + At + B \tag{9}$$

with

$$p = 0, \qquad K > 0, \qquad B > 0, \qquad A^2 < 4KB, \tag{10}$$

or

$$F(t) = -2Kp^{-1} + A\exp(t\sqrt{p}) + B\exp(-t\sqrt{p}) \tag{11}$$

with

$$p > 0, \qquad A > 0, \qquad B > 0, \qquad K^2 < p^2 AB, \tag{12}$$

or

$$F(t) = -2Kp^{-1} + A\cos(t\sqrt{-p}) + B\sin(t\sqrt{-p}) \tag{13}$$

with

$$p < 0, \qquad K > 0, \qquad p^2(A^2 + B^2) < 4K^2, \tag{14}$$

p being the constant occurring in (8) *(where $\varphi = F$). In particular, F has a positive analytic extension from I to the whole of $\mathbb{R}$.*

(iii) *Conversely, if N is a 3-dimensional manifold of constant sectional curvature K and F is given by* (9) *(resp.* (11) *or* (13)*) with* (10) *(resp.* (12) *or* (14)*), then the warped product $M = \mathbb{R} \times_F N$ has harmonic curvature, its Ricci tensor S is not parallel and the number of distinct eigenvalues of S does not exceed two at any point.*

Proof. (i) By Lemma 3 and (i) of Lemma 2, the tangent bundle of a neighbourhood of x splits as the orthogonal direct sum of the eigendistributions of S, i.e., of a geodesic line field V_λ and an integrable codimension one distribution V_μ with totally umbilic leaves, each of constant mean curvature. In a suitable local chart $x^0, x^1, \ldots, x^{n-1}$ with $\partial/\partial x^0 \in V_\lambda$, $\partial/\partial x^i \in V_\mu$ (i, j range in the sequel over $1, \ldots, n-1$), we have $g_{0i}=0$. Since V_λ is geodesic, $\Gamma^i_{00}=0$, i.e., $\partial_i g_{00}=0$. As V_μ is totally umbilic, the second fundamental form of its leaves is a multiple of the metric, that is, $-\Gamma^0_{ij} g_{00} = \zeta g_{ij}$, ζ being the mean curvature function, which yields $\partial_0 g_{ij} = 2 g_{00}^{-1} \zeta g_{ij}$. The constancy of ζ along V_μ says now that $\partial_0 g_{ij}(x^0, \ldots, x^{n-1}) = f(x^0) g_{ij}(x^0, \ldots, x^{n-1})$ for some function f of a real variable. Choosing a function q with $q'=f$, we have $\partial_0(e^{-q} g_{ij})=0$, i.e., $g_{ij}(x^0, \ldots, x^{n-1}) = e^{q(x^0)} h_{ij}(x^1, \ldots, x^{n-1})$ for some h_{ij}. Thus, a neighbourhood U of x can be chosen isometric to a warped product $I \times_F V$, I being an interval of $\mathbb{R}$, so that our decomposition of TU corresponds to the product decomposition of $T(I \times V)$. The function F is not constant, for otherwise U would just be the Riemannian product of I and V, the latter being an Einstein space, since its Ricci tensor has only one eigenvalue. Consequently, we would have $\nabla S = 0$ in U, which is a contradiction. Our assertion is now immediate from Lemma 4.

(ii) The three-dimensional Einstein space V is a space of constant curvature $K = \frac{\kappa}{6}$. On the other hand, for $n=4$ equation (8) takes the particularly simple form

$$F'' - pF = 2K \tag{15}$$

and its solutions are given by (9), (11) or (13) with $p=0$, $p>0$ or $p<0$, respectively, A and B being arbitrary.

If $\lambda_1, \ldots, \lambda_4$ denote the eigenvalues of S at any point, the second elementary symmetric function $\sigma_2(S) = \sum_{a<\beta} \lambda_a \lambda_\beta$ is well-defined and analytic everywhere on M. Using the identification $U = I \times_F V$ as in (i) and the notations of Remark 1, we have

$$\sigma_2(S) = 3F^{-4}[p^2 F^4 - pKF^3 + \tfrac{1}{4} p (F')^2 F^2 - 2K^2 F^2 + K (F')^2 F - \tfrac{1}{8}(F')^4]. \tag{16}$$

For a fixed $y \in V$, the curve $I \ni t \mapsto (t, y) \in U$ is a geodesic (cf. Remark 1). Therefore the right-hand side of (16), being a priori just an analytic function on I, is equal to the composition of $\sigma_2(S)$ with a geodesic of M. Hence it must have an analytic extension to $\mathbb{R}$, as M is complete. This implies that for any $t \in \mathbb{R}$ with $F(t)=0$, we have $F'(t)=0$. The solutions of (15) for which such a t exists are given by (9) with $A^2 = 4KB$ (resp. by (11) or (13) with $p^2 AB = K^2$ or $p^2(A^2+B^2) = 4K^2$). Substituting $q = \log F$ into (6), one sees easily that these conditions imply $\nabla S = 0$, which is a contradiction. Therefore F has no zeros in $\mathbb{R}$, i.e., $F>0$ everywhere. According to whether p is zero, positive or negative, this is equivalent to (10), (12) or (14), as required.

(iii) Since F is a solution of (8) with $n=4$, Lemma 4 implies harmonicity of the curvature tensor of M. From (5) together with $\rho_{ij} = 2Kh_{ij} = 2Ke^{-q} g_{ij}$ (notation of Remark 1) we obtain $E_S \leqq 2$ everywhere. Computing $\nabla_0 S_{00}$ from (6) with $q = \log F$, one sees that $\nabla S \neq 0$. This completes the proof.

Corollary 1. *Let (M, g) be a four-dimensional analytic Riemannian manifold with harmonic curvature tensor. If the number of distinct eigenvalues of the Ricci tensor S does not exceed 2 at any point and S is not parallel, then M is conformally flat.*

Proof. By (i) of Theorem 1, M contains an open subset U isometric to $I \times_F V$, where I is an interval and V is a three-dimensional space of constant curvature. It is easy to verify that for any such warped product the Weyl tensor $W=0$, which completes the proof.

4. Global Classification Theorems

Let $M_{K,A,B,p}$ denote the four-dimensional warped product manifold $\mathbb{R} \times_F N_K$, where N_K is the complete simply connected three-dimensional Riemannian manifold of constant sectional curvature K and the positive function $F\colon \mathbb{R} \to \mathbb{R}$ is given by (9) or (11) or (13), the real numbers K, A, B, p being respectively assumed to satisfy (10) or (12) or (14).

Lemma 5. *Let (M, g), (N, h) be complete Riemannian manifolds, $F\colon M \to \mathbb{R}$ a positive function. If F is bounded from below by a positive constant, then $M \times_F N$ is complete.*

Proof. Suppose $F \geqq c > 0$. For a vector $X \in T(M \times_F N)$ we have $(g \times_F h)(X, X) \geqq (g \times ch)(X, X)$. Therefore the distance functions satisfy $d_{g \times_F h} \geqq d_{g \times ch}$ and our assertion follows from completeness of the product metric $g \times ch$.

Theorem 2. *Let M be a complete, simply connected, analytic, four-dimensional Riemannian manifold with harmonic curvature and non-parallel Ricci tensor S. If S has less than three distinct eigenvalues at any point of (a non-void open subset of) M, then M is isometric to one of the manifolds $M_{K,A,B,p}$ as described above. Conversely, each of these manifolds has the properties just stated.*

Proof. It is immediate from Theorem 1 that a certain non-void connected open subset of M is isometric to an open subset of some $M_{K,A,B,p}$. Moreover, $M_{K,A,B,p}$ is complete in view of Lemma 5. Our assertion follows now from the extension theorem for analytic isometries ([4], p. 252) and (iii) of Theorem 1.

Lemma 6. (i) *For $p \geqq 0$, the Riemannian manifolds $M_{K,A,B,p}$ do not cover isometrically any compact manifold.*

(ii) *For $p < 0$, the set of all fixed-point-free, orientation preserving isometries of $M_{K,A,B,p}$ is contained in the direct product $G_p \times SO(4)$, acting on the underlying manifold $\mathbb{R} \times S^3$ in the product manner. Here $G_p \subset \mathbb{R}$ is the group of all translations of $\mathbb{R}$ leaving F invariant, i.e., $G_p = 2\pi(-p)^{-1/2}\mathbb{Z}$.*

Proof. From (5) together with $\rho_{ij} = 2Kh_{ij}$ (cf. Remark 1) it follows that both natural distributions of the underlying manifold $\mathbb{R} \times N_K$ are invariant under any isometry of $M_{K,A,B,p}$, as they are the eigendistributions of the Ricci tensor. Therefore every isometry is a Cartesian product $\theta \times \eta$ of a diffeomorphism of $\mathbb{R}$ with a diffeomorphism of N_K. The definition of warped product yields $(\theta \times \eta)^*(g \times_F h) = \theta^* g \times_{F \circ \theta} \eta^* h$, g and h being the metrics of $\mathbb{R}$ and N_K, respectively. Hence $\theta^* g = g$, $F \circ \theta = \tau F$ and $\eta^* h = \tau^{-1} h$ for some positive real number τ. From (9),

(11) or (13) it follows immediately that $\tau=1$. Thus, F is invariant under any isometry of $M_{K,A,B,p}$, i.e., it projects to a function on every isometric quotient. The unboundedness of F when $p \geqq 0$ implies now (i). Assume now that $\theta \times \eta$ is a fixed-point-free and orientation preserving isometry of $M_{K,A,B,p}$, $p<0$. If both θ and η reversed the orientation, $\theta \times \eta$ would have a fixed point, since η would belong to $O(4)-SO(4)$. Therefore θ is a translation and $\eta \in SO(4)$, which completes the proof.

Remark 3. Given a Ricci-flat manifold N of arbitrary dimension $n-1$, $n \geqq 3$, there exist positive functions $F\colon \mathbb{R} \to \mathbb{R}$ such that the warped products $\mathbb{R} \times_F N$ have harmonic curvature and non-parallel Ricci tensor. These functions are given by $F(t)=[A\exp(t\sqrt{p})+B\exp(-t\sqrt{p})]^{4/n}$, A, B and p being positive real parameters (cf. Lemma 4 and proof of Theorem 1). However, none of these warped products admits compact isometric quotients. In fact, one can verify as in the proof of Lemma 6 that the unbounded function F is invariant under any isometry of $\mathbb{R} \times_F N$.

Given real numbers K, A, B, p satisfying (14), a positive integer m and an orientation preserving isometry $Q \in SO(4)$ of $S^3 = N_K$, let us define the Riemannian manifold $M = M_{K,A,B,p,m,Q}$ as the quotient $M = M_{K,A,B,p}/\Gamma_{m,Q,p}$, $\Gamma_{m,Q,p}$ being the infinite cyclic group of isometries generated by $(2m\pi(-p)^{-1/2}, Q)$, in the notation of Lemma 6. It is clear that $M_{K,A,B,p,m,Q}$ is a compact Riemannian manifold, diffeomorphic to $S^1 \times S^3$. By Theorem 2, it has harmonic curvature, while its Ricci tensor is not parallel and has less than three eigenvalues at any point.

The manifolds $M_{K,A,B,p,m,Q}$ have the following property of universality:

Theorem 3. *Let (M,g) be a four-dimensional compact analytic Riemannian manifold with harmonic curvature. If the Ricci tensor of M is not parallel and the number of its distinct eigenvalues does not exceed two at any point of M, then M is covered isometrically by one of the manifolds $M_{K,A,B,p,m,Q}$ defined above.*

Proof. We may assume that M is orientable. Thus, $M = M_{K,A,B,p}/\Gamma$, where, in view of Theorem 2 and Lemma 6, K, A, B, p satisfy (14) and $\Gamma \subset G_p \times SO(4)$ is a discrete subgroup. If Γ were contained in $\{0\} \times SO(4)$, M would not be compact. Therefore Γ contains $\Gamma_{m,Q,p}$ for some $Q \in SO(4)$ and some positive integer m, which completes the proof.

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